

Half-Positional Objectives Recognized by Deterministic Büchi Automata

Antonio Casares

LaBRI, Université de Bordeaux

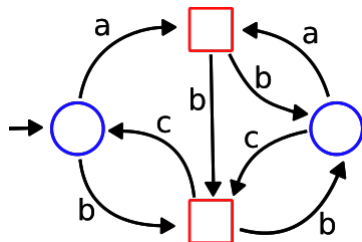
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Joint work with Patricia Bouyer, Mickael Randour and Pierre Vandenhove

Games and positionality

Games over graphs

Arena: oriented graph $\mathcal{G} = (V = (V_{\text{Eve}} \uplus V_{\text{Adam}}), E, v_0)$ with edges labeled by colors in a set C and an initial vertex.



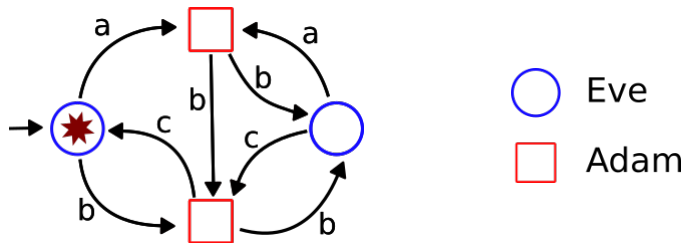
○ Eve

□ Adam

$C = \{a, b, c\}$

Players move a token in turns producing an infinite word $w \in C^\omega$.

Games over graphs

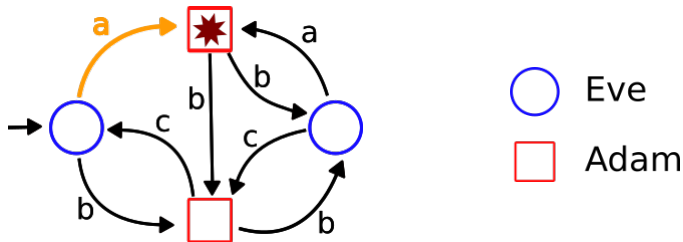


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Games over graphs

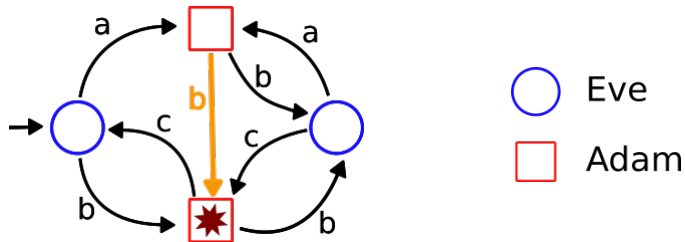


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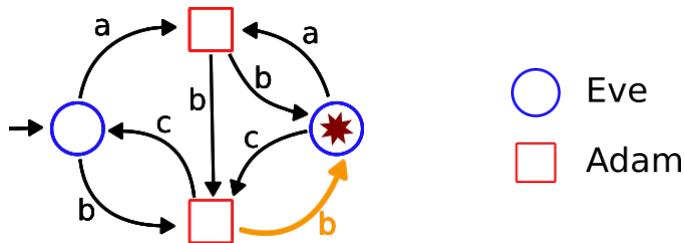


Players move a token in turns producing an infinite word

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$$\text{Output} = ab$$

Games over graphs

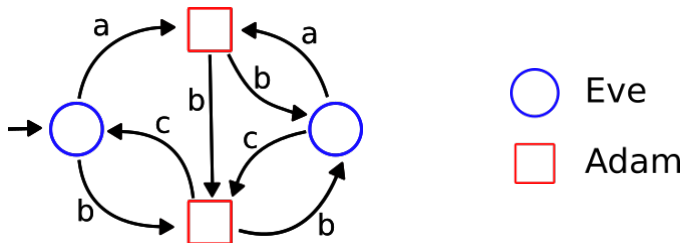


Players move a token in turns producing an infinite word

$$w \in C^\omega.$$

$$\text{Output} = abb\dots$$

Games over graphs



Objective: Set $\mathbb{W} \subseteq C^\omega$ of winning sequences.

Eve wins a play if $w \in \mathbb{W}$.

Adam wins a play if $w \notin \mathbb{W}$.

\mathbb{W} -game = Arena + Objective \mathbb{W}

Strategy (for Eve)

Function $\sigma : E^* \times V_{\text{Eve}} \rightarrow E$ prescribing how Eve should move depending on the past of the play.

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Positional strategy

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(Eve's choices depend exclusively on the current position).

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(Eve's choices depend exclusively on the current position).

Winner

We say that Eve *wins* a \mathbb{W} -game \mathcal{G} if she has a strategy σ such that all paths from v_0 consistent with that strategy belong to \mathbb{W} .

Positional objective

An objective $\mathbb{W} \subseteq C^\omega$ is *half-positional*¹ if for every \mathbb{W} -game \mathcal{G} Eve has a positional strategy σ such that

$$\text{Eve wins } \mathcal{G} \implies \text{Eve wins } \mathcal{G} \text{ using } \sigma.$$

¹In this talk positional = half-positional.

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Bi-positional objective

An objective $\mathbb{W} \subseteq C^\omega$ is *bi-positional* if both \mathbb{W} and $C^\omega \setminus \mathbb{W}$ are half-positional.

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Positional objective

An objective $\mathbb{W} \subseteq C^\omega$ is *half-positional*¹ **over finite arenas** if for every **finite** \mathbb{W} -game \mathcal{G} Eve has a positional strategy σ such that

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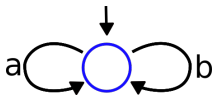
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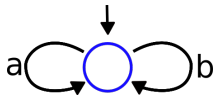
Examples

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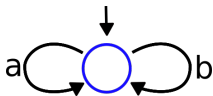


- ▶ $\mathbb{W} = C^*a^4C^\omega$

?

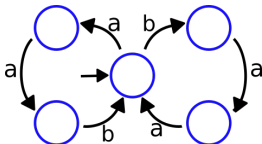
Examples

- ▶ $\mathbb{W} = C^*(ab)^\omega$ is not positional.



- ▶ $\mathbb{W} = C^*a^4C^\omega$

Not positional



Example: Büchi condition

- ▶ For $F \subseteq C$ we define

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Lemma (Emerson, Jutla 1991)

$\mathbb{W} = \text{Büchi}(F)$ is *positional*.

▶ $\mathbb{W} = \text{Büchi}(a) \cup C^* a^2 C^\omega$.

?

▶ $\mathbb{W} = \text{Büchi}(a) \cup C^* a^2 C^\omega.$

You will know at the end of the talk!

Bi-positionality is quite well understood:

Some known results about bi-positionality

- ▶ Characterization of bi-positionality over finite arenas [Gimbert, Zielonka '05].
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But for applications in synthesis, half-positionality is more relevant!

Some known results about half-positionality

- ▶ Some sufficient conditions for half-positionality [Kopczyński '08, BFMM '10].
- ▶ Charact. of half-positional conditions over all arenas using universal graphs [Ohlmann '21].

Some known results about half-positionality

- ▶ Some sufficient conditions for half-positionality [Kopczyński '08, BFMM '10].
- ▶ **Characterization of half-positional conditions over all arenas using universal graphs [Ohlmann '21].**

→Structural characterization:

\mathbb{W} positional \Leftrightarrow There exists a suitable structure for any cardinal.

Not effective.

Open question

Effective characterization of positionality for ω -regular conditions.

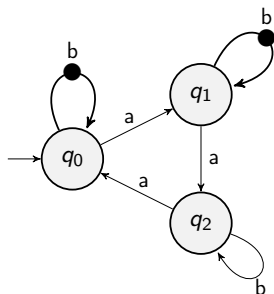
In this work:

Main result

Effective characterization of positionality for languages recognized by **deterministic Büchi automata**.

Deterministic Büchi automata

In this talk, all automata will be deterministic.



- Automaton \mathcal{B}
- $C = \{a, b\}$ input alphabet
- Input = $w \in C^\omega$
- $\overset{\bullet}{\rightarrow}$ = Büchi transition

- We accept a run if it visits infinitely often a Büchi transition.
- $\mathcal{L}(\mathcal{B}) = \{w \in C^\omega : \mathcal{B} \text{ has an accepting run over } w\}$.

Condition over transitions

Remark: Acceptance condition is defined over *transitions* of the automata.



DBA-recognizability

We say that an objective $\mathbb{W} \subseteq C^\omega$ is **DBA-recognizable** if there is a deterministic Büchi automaton \mathcal{B} such that

$$\mathbb{W} = \mathcal{L}(\mathcal{B}).$$

Remark: the class of DBA-recognizable objectives is a proper subclass of ω -regular objectives.

ω -regular = Recognizable by ND-Büchi = Recognizable by det. parity.

We fix an objective $\mathbb{W} \subseteq C^\omega$.

For a finite word $u \in C^*$ we write

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For $u, v \in C^*$:

$u \prec v$ if $u^{-1}\mathbb{W} \subsetneq v^{-1}\mathbb{W}$ (Preorder).

$u \sim v$ if $u^{-1}\mathbb{W} = v^{-1}\mathbb{W}$ (Equivalence relation).

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If \mathbb{W} is recognized by a Büchi automaton \mathcal{B} , we can define analogous relations between the states of \mathcal{B} .

Prefix-classifier $\mathcal{B}_{\sim} = (Q, q_0, \delta : Q \times C \rightarrow Q)$:

- ▶ $Q = \{u^{-1}\mathbb{W} : u \in C^*\}$.
- ▶ $q_0 = \varepsilon^{-1}\mathbb{W}$.
- ▶ $\delta(u^{-1}\mathbb{W}, c) = (uc)^{-1}\mathbb{W}$.

Automaton structure without acceptance condition.

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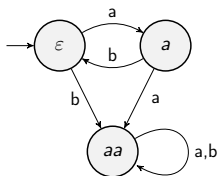
Automaton structure without acceptance condition.

Sometimes we can define a Büchi condition such that $\mathcal{L}(\mathcal{B}_{\sim}) = \mathbb{W}$, but not always (even if \mathbb{W} is DBA-recognizable)!!

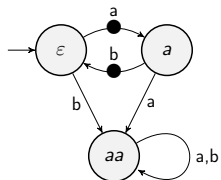
If it is possible to do so, we say that \mathbb{W} *can be recognized by its prefix-classifier*.

Examples

► $\mathbb{W} = (ab)^\omega$

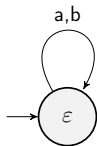


Prefix-classifier \mathcal{B}_\sim .



Büchi condition so that $\mathcal{L}(\mathcal{B}_\sim) = \mathbb{W}$.

► $\mathbb{W} = (C^*aa)^\omega$



Prefix-classifier \mathcal{B}_\sim .

We cannot define a Büchi condition over \mathcal{B}_\sim so that $\mathcal{L}(\mathcal{B}_\sim) = \mathbb{W}$.

Example: prefix-independent conditions

\mathbb{W} is *prefix-independent* if $\forall w \in C^\omega, \forall u \in C^*$

$$w \in \mathbb{W} \Leftrightarrow uw \in \mathbb{W}.$$

Equivalently, if \mathcal{B}_\sim has a single state.

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Lemma

If \mathbb{W} is *prefix-independent*, \mathbb{W} can be recognized by its *prefix-classifier* if and only if

$$\mathbb{W} = \text{Büchi}(F) \text{ for some } F \subseteq C.$$

Remember: Transition based acceptance!

Three sufficient and necessary conditions for half-positionality

Condition 1: \prec is a total order

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Prefix preorder \prec is total.

Lemma (Necessity of Condition 1)

If \prec is not total, \mathbb{W} is not positional over finite arenas.

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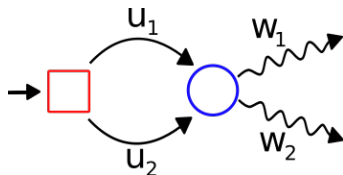
Lemma (Necessity of Condition 1)

If \prec is not total, \mathbb{W} is not positional over finite arenas.

Proof: If \prec is not total, there are $u_1, u_2 \in C^*$ and $w_1, w_2 \in C^\omega$ such that:

$$u_1 w_1 \in \mathbb{W}, \quad u_2 w_1 \notin \mathbb{W},$$

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Remark: we can make it a finite arena using ω -regularity.

Condition 2: Progress-consistency

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We say that \mathbb{W} is *progress-consistent* if for all $u, v \in C^*$:

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Lemma (Necessity of Condition 2)

If \mathbb{W} is not progress-consistent, it is not positional over finite arenas.

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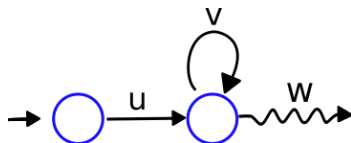
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Lemma (Necessity of Condition 2)

If \mathbb{W} is not progress-consistent, it is not positional over finite arenas.

Proof: Let u, v such that $u \prec uv$ and $uv^\omega \notin \mathbb{W}$. There is $w \in C^\omega$ s.t. $uw \notin \mathbb{W}$ but $uvw \in \mathbb{W}$.



Remark: Necessity of Conditions 1 and 2 apply to all objectives (over infinite arenas), not necessarily ω -regular.

Condition 3: Recognizability by the prefix-classifier

Condition 3

Objective \mathbb{W} is recognizable by a Büchi automaton built on top of the prefix-classifier \mathcal{B}_{\sim} .

Lemma (Necessity of Condition 3)

If \mathbb{W} is not recognizable by its prefix-classifier \mathcal{B}_{\sim} , it is not positional over finite arenas.

Condition 3: Recognizability by the prefix-classifier

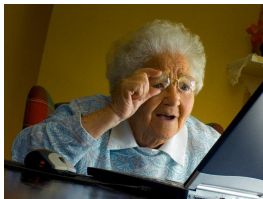
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Lemma (Necessity of Condition 3)

If \mathbb{W} is not recognizable by its prefix-classifier \mathcal{B}_{\sim} , it is not positional over finite arenas.

Proof: Quite technical.



Condition 3: Recognizability by the prefix-classifier

Proof outline: \mathbb{W} an objective recognized by a DBA \mathcal{B} .

- 1 Prefix-independent case.

Lemma

If $|\mathcal{B}_{\sim}| = 1$ (\mathbb{W} prefix-independent), then

\mathbb{W} positional $\Leftrightarrow \mathbb{W} = \text{Büchi}(F)$ for some $F \subseteq C$.

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If $|\mathcal{B}_{\sim}| = 1$ (\mathbb{W} prefix-independent), then

$$\mathbb{W} \text{ positional} \Leftrightarrow \mathbb{W} = \text{Büchi}(F) \text{ for some } F \subseteq C.$$

- 2 General case.

We reduce to the prefix independent case. For each equivalent class of states $[q]$, $q \in \mathcal{B}$ we consider the alphabet

$$C_{[q]} = \{u \in C^* : \delta(q, u) \in [q]\}.$$

and the prefix-independent winning condition

$$\mathbb{W}_{[q]} = \{w \in C_{[q]}^\omega : w \in q^{-1}\mathbb{W}\}.$$

Using the previous lemma, we show that if \mathbb{W} is positional we can merge all states in $[q]$.

Conditions for half-positionality

$\mathbb{W} \subseteq C^\omega$ a DBA-recognizable objective.

- ▶ Prefix preorder \preceq is total.
- ▶ Progress-consistency.
- ▶ Recognizability by the prefix-classifier \mathcal{B}_\sim .

Proposition (Necessity of the conditions)

If a DBA-recognizable objective $\mathbb{W} \subseteq C^\omega$ is half-positional over finite arenas, then it verifies the three previous conditions.

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They are also sufficient (over all arenas)!

Sufficiency of the conditions

The main ingredient to prove the sufficiency of the conditions is:

Theorem (Ohlmann 2021)

An objective $\mathbb{W} \subseteq C^\omega$ is half-positional if and only if for every cardinal κ there exists a (\mathbb{W}, κ) -universal well-monotonic graph \mathcal{U} .

(\mathbb{W}, κ) -universal well-monotonic graph

(\mathcal{U}, \leq) edge-labeled graph with a well order such that:

$\mathcal{U} \models \mathbb{W}$: All paths of \mathbb{W} satisfy \mathbb{W} .

Well-monotonic: edges in \mathcal{U} “are closed by the order relation”.

(\mathbb{W}, κ) -universal: If \mathcal{G} is another graph such that all its paths satisfy \mathbb{W} and $|\mathcal{G}| < \kappa$, then it can be embedded in \mathcal{U} .

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The existence of such graphs is a structural witness of positionality.

Proposition

If $\mathbb{W} \subseteq C^\omega$ is a DBA-recognizable objective verifying

- ▶ Prefix preorder \preceq is total,
- ▶ Progress-consistency,
- ▶ Recognizability by the prefix-classifier \mathcal{B}_{\sim} ,

then, there is a (\mathbb{W}, κ) -universal well-monotonic graph for every cardinal κ .

Corollary (Sufficiency of the conditions)

If a DBA-recognizable objective $\mathbb{W} \subseteq C^\omega$ verifies the three previous conditions, then it is half-positional.

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Theorem

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Corollary (Complexity)

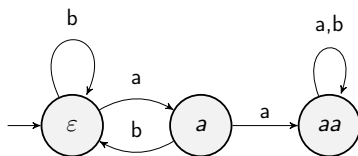
Given a Büchi automaton \mathcal{B} , we can determine in polynomial time whether $\mathcal{L}(\mathcal{B})$ is half-positional.

Corollary (1-to-2 players lift)

Let \mathbb{W} be a DBA-recognizable objective. If \mathbb{W} is positional over finite one-player arenas, then it is half-positional over all arenas (2 players and of any cardinality).

Examples

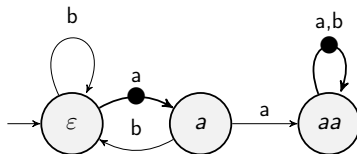
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Prefix-classifier \mathcal{B}_\sim .

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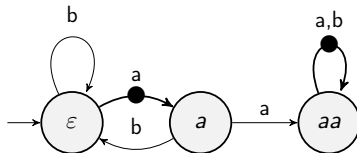
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Büchi condition on top of the prefix-classifier \mathcal{B}_\sim .

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Büchi condition on top of the prefix-classifier \mathcal{B}_\sim .

Positional!

Remark: $\mathbb{W} = \text{Büchi}(a) \cup C^* a^2 C^\omega$ is not bi-positional:

$$C^\omega \setminus \mathbb{W} = (b^* ab)^* b^\omega \text{ (Not progress-consistent).}$$

Open questions

Characterization of positionality for ω -regular languages.

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Subquestions:

- ▶ Characterization of positionality for languages recognized by deterministic co-Büchi automata.
- ▶ Union prefix-independent positional ω -regular conditions is positional?
- ▶ For ω -regular conditions, positionality over finite arenas implies positionality over arbitrary arenas?
- ▶ 1-to-2 players lift for ω -regular conditions.

Characterization of positionality for ω -regular languages.

Subquestions:

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Thank you!