Half-Positional Objectives Recognized by Deterministic Büchi Automata

Antonio Casares

LaBRI, Université de Bordeaux

30 May 2022

Joint work with Patricia Bouyer, Mickael Randour and Pierre Vandenhove
Games and positionality
**Arena:** oriented graph \( \mathcal{G} = (V = (V_{\text{Eve}} \cup V_{\text{Adam}}), E, v_0) \) with edges labeled by colors in a set \( C \) and an initial vertex.

Players move a token in turns producing an infinite word \( w \in C^\omega \).
Games over graphs

Players move a token in turns producing an infinite word $w \in C^\omega$.

Output =
Games over graphs

Players move a token in turns producing an infinite word \( w \in C^\omega \).

Output = a
Games over graphs

Players move a token in turns producing an infinite word $w \in C^\omega$.

Output = $ab$
Games over graphs

Players move a token in turns producing an infinite word $w \in C^\omega$.

Output = $abb \ldots$
Games over graphs

Objective: Set $W \subseteq C^\omega$ of winning sequences.

Eve wins a play if $w \in W$.
Adam wins a play if $w \notin W$.

$W$-game = Arena + Objective $W$
Strategy (for Eve)

Function $\sigma : E^* \times V_{Eve} \rightarrow E$ prescribing how Eve should move depending on the past of the play.
## Strategies

### Strategy (for Eve)

Function $\sigma : E^* \times V_{Eve} \rightarrow E$ prescribing how Eve should move depending on the past of the play.

### Positional strategy

Function

$$\sigma : V_{Eve} \rightarrow E,$$

(Eve’s choices depend exclusively on the current position).
**Strategy (for Eve)**

Function $\sigma : E^* \times \mathcal{V}_{Eve} \rightarrow E$ prescribing how Eve should move depending on the past of the play.

**Positional strategy**

Function

$$\sigma : \mathcal{V}_{Eve} \rightarrow E,$$

(Eve’s choices depend exclusively on the current position).

**Winner**

We say that Eve wins a $\mathbb{W}$-game $G$ if she has a strategy $\sigma$ such that all paths from $v_0$ consistent with that strategy belong to $\mathbb{W}$.
Positionality

Positional objective

An objective $\mathbb{W} \subseteq C^\omega$ is half-positional\(^1\) if for every $\mathbb{W}$-game $\mathcal{G}$ Eve has a positional strategy $\sigma$ such that

$$\text{Eve wins } \mathcal{G} \implies \text{Eve wins } \mathcal{G} \text{ using } \sigma.$$ 

\(^1\)In this talk positional = half-positional.
Positionality

**Positional objective**

An objective $\mathbb{W} \subseteq C^\omega$ is *half-positional*\(^1\) if for every $\mathbb{W}$-game $\mathcal{G}$ Eve has a positional strategy $\sigma$ such that

$$\text{Eve wins } \mathcal{G} \implies \text{Eve wins } \mathcal{G} \text{ using } \sigma.$$  

\(^{1}\)In this talk positional = half-positional.

**Bi-positional objective**

An objective $\mathbb{W} \subseteq C^\omega$ is *bi-positional* if both $\mathbb{W}$ and $C^\omega \setminus \mathbb{W}$ are half-positional.
Positionality over finite arenas

Positional objective

An objective \( W \subseteq C^\omega \) is half-positional\(^1\) over finite arenas if for every finite \( W \)-game \( G \) Eve has a positional strategy \( \sigma \) such that

\[
\text{Eve wins } G \implies \text{Eve wins } G \text{ using } \sigma.
\]

Bi-positional objective

An objective \( W \subseteq C^\omega \) is bi-positional over finite arenas if both \( W \) and \( C^\omega \setminus W \) are half-positional over finite arenas.

\(^1\)In this talk positional = half-positional.
Examples

$W = C^*(ab)^\omega$ is not positional.
Examples

▶ $W = C^*(ab)^\omega$ is not positional.

▶ $W = C^*a^4C^\omega$
Examples

▶ $W = C^*(ab)\omega$ is not positional.

▶ $W = C^*a^4C\omega$

Not positional
For $F \subseteq C$ we define

$$\text{Büchi}(F) = \{ w \in C^\omega : \text{some } c \in F \text{ appears infinitely often in } w \}.$$
For $F \subseteq C$ we define

$$\text{B"uch}(F) = \{ w \in C^\omega : \text{some } c \in F \text{ appears infinitely often in } w \}.$$
Examples

\( W = \text{Büchi}(a) \cup C^* a^2 C^\omega. \)
Examples

\[ W = \text{B"{u}chi}(a) \cup C^* a^2 C^\omega. \]

You will know at the end of the talk!
Bi-positionality is quite well understood:

- Characterization of bi-positionality over finite arenas [Gimbert, Zielonka ’05].
- Characterization of bi-positionality over all arenas [Colcombet, Niwiński ’06].
Bi-positionality is quite well understood:

- Characterization of bi-positionality over finite arenas [Gimbert, Zielonka ’05].
- Characterization of bi-positionality over all arenas [Colcombet, Niwiński ’06].

But for applications in synthesis, half-positionality is more relevant!
Some known results about half-positionality

- Some sufficient conditions for half-positionality [Kopczyński ’08, BFMM ’10].
- Charact. of half-positional conditions over all arenas using universal graphs [Ohlmann ’21].
Some known results

Some known results about half-positionality

- Some sufficient conditions for half-positionality [Kopczyński ’08, BFMM ’10].
- Characterization of half-positional conditions over all arenas using universal graphs [Ohlmann ’21].

Structural characterization:

\[ \forall \text{positional } \iff \text{There exists a suitable structure for any cardinal.} \]

Not effective.
Contribution

Open question

Effective characterization of positionality for $\omega$-regular conditions.

In this work:

Main result

Effective characterization of positionality for languages recognized by deterministic Büchi automata.
Deterministic Büchi automata
In this talk, all automata will be deterministic.

Automaton $\mathcal{B}$

- $C = \{a, b\}$ input alphabet
- Input $= w \in C^\omega$
- $\Rightarrow = \text{Büchi transition}$

We accept a run if it visits infinitely often a Büchi transition.

$\mathcal{L}(\mathcal{B}) = \{w \in C^\omega : \mathcal{B} \text{ has an accepting run over } w\}$. 
Remark: Acceptance condition is defined over transitions of the automata.
Recognizability by Büchi automata

DBA-recognizability

We say that an objective $W \subseteq C^\omega$ is **DBA-recognizable** if there is a deterministic Büchi automaton $B$ such that

$$W = \mathcal{L}(B).$$

**Remark:** the class of DBA-recognizable objectives is a proper subclass of $\omega$-regular objectives.

$\omega$—regular $=$ Recognizable by ND-Büchi $=$ Recognizable by det. parity.
We fix an objective $W \subseteq C^\omega$.

For a finite word $u \in C^*$ we write

$$u^{-1}W = \{w \in C^\omega : uw \in W\}.$$
We fix an objective $W \subseteq C^\omega$.

For a finite word $u \in C^*$ we write

$$u^{-1}W = \{ w \in C^\omega : uw \in W \}.$$

For $u, v \in C^*$:

- $u \prec v$ if $u^{-1}W \subsetneq v^{-1}W$ (Preorder).
- $u \sim v$ if $u^{-1}W = v^{-1}W$ (Equivalence relation).
Right congruence

We fix an objective $\mathcal{W} \subseteq C^\omega$.

For a finite word $u \in C^*$ we write

$$u^{-1}\mathcal{W} = \{w \in C^\omega : uw \in \mathcal{W}\}.$$

For $u, v \in C^*$:

- $u \prec v$ if $u^{-1}\mathcal{W} \subsetneq v^{-1}\mathcal{W}$ (Preorder).
- $u \sim v$ if $u^{-1}\mathcal{W} = v^{-1}\mathcal{W}$ (Equivalence relation).

If $\mathcal{W}$ is recognized by a Büchi automaton $\mathcal{B}$, we can define analogous relations between the states of $\mathcal{B}$. 
Prefix-classifier $B_\sim = (Q, q_0, \delta : Q \times C \rightarrow Q)$:

- $Q = \{ u^{-1}W : u \in C^* \}$.
- $q_0 = \varepsilon^{-1}W$.
- $\delta(u^{-1}W, c) = (uc)^{-1}W$.

Automaton structure without acceptance condition.
Prefix-classifier \( B_\sim = (Q, q_0, \delta : Q \times C \rightarrow Q) \):

- \( Q = \{ u^{-1}W : u \in C^* \} \).
- \( q_0 = \varepsilon^{-1}W \).
- \( \delta(u^{-1}W, c) = (uc)^{-1}W \).

Automaton structure without acceptance condition.

Sometimes we can define a Büchi condition such that \( L(B_\sim) = W \), but not always (even if \( W \) is DBA-recognizable)!!

If it is possible to do so, we say that \( W \) can be recognized by its prefix-classifier.
Examples

\[ W = (ab)^\omega \]

Prefix-classifier \( B_\sim \).

Büchi condition so that \( \mathcal{L}(B_\sim) = W \).
Examples

\[ W = (C^* aa)^\omega \]

Prefix-classifier \( \mathcal{B}_\sim \).

We cannot define a Büchi condition over \( \mathcal{B}_\sim \) so that \( \mathcal{L}(\mathcal{B}_\sim) = W \).
\( \mathcal{W} \) is prefix-independent if \( \forall w \in C^\omega, \forall u \in C^* \)

\[ w \in \mathcal{W} \iff uw \in \mathcal{W}. \]

Equivalently, if \( \mathcal{B}_\sim \) has a single state.
Example: prefix-independent conditions

\[ \mathcal{W} \text{ is prefix-independent if } \forall w \in C^\omega, \forall u \in C^* \]

\[ w \in \mathcal{W} \iff uw \in \mathcal{W}. \]

Equivalently, if \( B\sim \) has a single state.

Lemma

If \( \mathcal{W} \) is prefix-independent, \( \mathcal{W} \) can be recognized by its prefix-classifier if and only if

\[ \mathcal{W} = \text{Büchi}(F) \text{ for some } F \subseteq C. \]

Remember: Transition based acceptance!
Three sufficient and necessary conditions for half-positionality
Condition 1: \( \prec \) is a total order

Prefix preorder \( \prec \) is total.

Lemma (Necessity of Condition 1)

*If \( \prec \) is not total, \( \mathbb{W} \) is not positional over finite arenas.*
Lemma (Necessity of Condition 1)

If $\prec$ is not total, $\mathcal{W}$ is not positional over finite arenas.

**Proof:** If $\prec$ is not total, there are $u_1, u_2 \in C^*$ and $w_1, w_2 \in C^\omega$ such that:

- $u_1 w_1 \in \mathcal{W}, \quad u_2 w_1 \notin \mathcal{W}$,
- $u_2 w_2 \in \mathcal{W}, \quad u_1 w_2 \notin \mathcal{W}$.

**Remark:** we can make it a finite arena using $\omega$-regularity.
We say that $W$ is *progress-consistent* if for all $u, v \in C^*$:

$$u \prec uv \implies uv^\omega \in W.$$
Condition 2: Progress-consistency

Condition 2

We say that \( \mathbb{W} \) is progress-consistent if for all \( u, v \in C^* \):

\[ u \prec uv \implies uv^\omega \in \mathbb{W}. \]

Lemma (Necessity of Condition 2)

If \( \mathbb{W} \) is not progress-consistent, it is not positional over finite arenas.
Condition 2: Progress-consistency

We say that $\mathbb{W}$ is *progress-consistent* if for all $u, v \in C^*$:

$$u \prec uv \implies uv^\omega \in \mathbb{W}.$$ 

Lemma (Necessity of Condition 2)

*If $\mathbb{W}$ is not progress-consistent, it is is not positional over finite arenas.*

**Proof:** Let $u, v$ such that $u \prec uv$ and $uv^\omega \notin \mathbb{W}$. There is $w \in C^\omega$ s.t. $uw \notin \mathbb{W}$ but $uvw \in \mathbb{W}$. 
Remark: Necessity of Conditions 1 and 2 apply to all objectives (over infinite arenas), not necessarily $\omega$-regular.
Condition 3: Recognizability by the prefix-classifier

**Condition 3**

Objective $W$ is recognizable by a Büchi automaton built on top of the prefix-classifier $B_{\sim}$.

**Lemma (Necessity of Condition 3)**

If $W$ is not recognizable by its prefix-classifier $B_{\sim}$, it is not positional over finite arenas.
Condition 3: Recognizability by the prefix-classifier

Objective $W$ is recognizable by a Büchi automaton built on top of the prefix-classifier $B_\sim$.

Lemma (Necessity of Condition 3)

If $W$ is not recognizable by its prefix-classifier $B_\sim$, it is not positional over finite arenas.

Proof: Quite technical.
Condition 3: Recognizability by the prefix-classifier

**Proof outline:** \( \mathcal{W} \) an objective recognized by a DBA \( \mathcal{B} \).

1. Prefix-independent case.

**Lemma**

If \( |\mathcal{B}| = 1 \) (\( \mathcal{W} \) prefix-independent), then

\[
\mathcal{W} \text{ positional } \iff \mathcal{W} = \text{Büchi}(F) \text{ for some } F \subseteq C.
\]
Condition 3: Recognizability by the prefix-classifier

**Proof outline:** \( W \) an objective recognized by a DBA \( B \).

1. Prefix-independent case.

**Lemma**

If \( |B_\sim| = 1 \) (\( W \) prefix-independent), then

\[ W \text{ positional } \Leftrightarrow W = \text{B"uchi}(F) \text{ for some } F \subseteq C. \]

2. General case.
   We reduce to the prefix independent case. For each equivalent class of states \([q], q \in B\) we consider the alphabet
   \[ C_{[q]} = \{ u \in C^* : \delta(q, u) \in [q] \}. \]
   and the prefix-independent winning condition
   \[ W_{[q]} = \{ w \in C_{[q]}^\omega : w \in q^{-1}W \}. \]
   Using the previous lemma, we show that if \( W \) is positional we can merge all states in \([q]\).
Necessity of the conditions

### Conditions for half-positionality

\( \mathbb{W} \subseteq C^\omega \) a DBA-recognizable objective.

- Prefix preorder \( \preceq \) is total.
- Progress-consistency.
- Recognizability by the prefix-classifier \( \mathcal{B}_\sim \).

### Proposition (Necessity of the conditions)

*If a DBA-recognizable objective \( \mathbb{W} \subseteq C^\omega \) is half-positional over finite arenas, then it verifies the three previous conditions.*
Sufficiency of the conditions

Conditions for half-positionality

\( \mathcal{W} \subseteq \omega \) a DBA-recognizable objective.
- Prefix preorder \( \preceq \) is total.
- Progress-consistency.
- Recognizability by the prefix-classifier \( \mathcal{B} \sim \).

They are also sufficient (over all arenas)!
Sufficiency of the conditions

The main ingredient to prove the sufficiency of the conditions is:

**Theorem (Ohlmann 2021)**

An objective $\mathbb{W} \subseteq C^\omega$ is half-positional if and only if for every cardinal $\kappa$ there exists a $(\mathbb{W}, \kappa)$-universal well-monotonic graph $\mathcal{U}$.

---

$(\mathbb{W}, \kappa)$-universal well-monotonic graph

$(\mathcal{U}, \leq)$ edge-labeled graph with a well order such that:

$\mathcal{U} \models \mathbb{W}$: All paths of $\mathbb{W}$ satisfy $\mathbb{W}$.

**Well-monotonic**: edges in $\mathcal{U}$ “are closed by the order relation”.

$(\mathbb{W}, \kappa)$-universal: If $\mathcal{G}$ is another graph such that all its paths satisfy $\mathbb{W}$ and $|\mathcal{G}| < \kappa$, then it can be embedded in $\mathcal{U}$. 
Sufficiency of the conditions

The main ingredient to prove the sufficiency of the conditions is:

Theorem (Ohlmann 2021)

An objective $\mathbb{W} \subseteq C^\omega$ is half-positional if and only if for every cardinal $\kappa$ there exists a $(\mathbb{W}, \kappa)$-universal well-monotonic graph $\mathcal{U}$.

The existence of such graphs is a structural witness of positionality.
Sufficiency of the conditions

**Proposition**

If $\mathbb{W} \subseteq C^\omega$ is a DBA-recognizable objective verifying

- Prefix preorder $\preceq$ is total,
- Progress-consistency,
- Recognizability by the prefix-classifier $B_\sim$,

then, there is a $(\mathbb{W}, \kappa)$-universal well-monotonic graph for every cardinal $\kappa$.

**Corollary (Sufficiency of the conditions)**

If a DBA-recognizable objective $\mathbb{W} \subseteq C^\omega$ verifies the three previous conditions, then it is half-positional.
Main result

**Conditions for half-positionality**

\( W \subseteq C^\omega \) a DBA-recognizable objective.

- Prefix preorder \( \preceq \) is total.
- Progress-consistency.
- Recognizability by the prefix-classifier \( B_\sim \).

**Theorem**

A DBA-recognizable objective \( W \subseteq C^\omega \) is half-positional if and only if it verifies the three previous conditions.
Main result

### Conditions for half-positionality

\[ \mathcal{W} \subseteq \mathcal{C}^\omega \text{ a DBA-recognizable objective.} \]

- Prefix preorder \( \preceq \) is total.
- Progress-consistency.
- Recognizability by the prefix-classifier \( \mathcal{B}_\sim \).

### Theorem

A DBA-recognizable objective \( \mathcal{W} \subseteq \mathcal{C}^\omega \) is half-positional if and only if it verifies the three previous conditions.
Corollary (Complexity)

Given a Büchi automaton $\mathcal{B}$, we can determine in polynomial time whether $\mathcal{L}(\mathcal{B})$ is half-positional.

Corollary (1-to-2 players lift)

Let $\mathbb{W}$ be a DBA-recognizable objective. If $\mathbb{W}$ is positional over finite one-player arenas, then it is half-positional over all arenas (2 players and of any cardinality).
Examples
\[
\mathcal{W} = \text{Büchi}(a) \cup C^* a^2 C^\omega.
\]

Prefix-classifier \( \mathcal{B}_{\sim} \).
Example

\[ W = \text{Büchi}(a) \cup C^* a^2 C^\omega. \]

Büchi condition on top of the prefix-classifier \( B_{\sim} \).
Example

$W = \text{Büchi}(a) \cup C^* a^2 C^\omega$.

Büchi condition on top of the prefix-classifier $B_\sim$.

Positional!
Remark: \( W = \text{Büchi}(a) \cup C^* a^2 C^\omega \) is not bi-positional:

\[
C^\omega \setminus W = (b^*ab)^* b^\omega \quad \text{(Not progress-consistent)}.
\]
Open questions
Open questions

Characterization of positionality for $\omega$-regular languages.
Open questions

Characterization of positionality for $\omega$-regular languages.

Subquestions:

- Characterization of positionality for languages recognized by deterministic co-Büchi automata.
- Union prefix-independent positional $\omega$-regular conditions is positional?
- For $\omega$-regular conditions, positionality over finite arenas implies positionality over arbitrary arenas?
- 1-to-2 players lift for $\omega$-regular conditions.
Open questions

Characterization of positionality for $\omega$-regular languages.

Subquestions:

▶ Characterization of positionality for languages recognized by deterministic co-Büchi automata.

▶ Union prefix-independent positional $\omega$-regular conditions is positional?

▶ For $\omega$-regular conditions, positionality over finite arenas implies positionality over arbitrary arenas?

▶ 1-to-2 players lift for $\omega$-regular conditions.

Thank you!