

Dynamics on Games: Simulation-Based Techniques and Applications to Routing

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Outline

- 1 A brief review of strategic games
 - Nash equilibrium
 - Symmetric two-player games
- 2 Evolutionary game theory
 - Evolutionary Stable Strategy
 - The Replicator Dynamics
 - Other Dynamics
- 3 Games played on graphs
 - Two examples of dynamics
 - Relations that maintain termination
 - More realistic conditions
 - Application to interdomain routing

Prisoner's dilemma

Prisoners' dilemma

Prisoners' dilemma		prisoner B			
		confess 	remain silent 		
prisoner A	confess 	 5 years	 5 years	 0 year	 20 years
	remain silent 	 20 years	 0 year	 1 year	 1 year

Prisoner's dilemma

Equivalently (since only the relative order of payoffs matters):

	C	S
C	(3, 3)	(1, 4)
S	(4, 1)	(2, 2)

The point of view of strategic games

	C	S
C	(3, 3)	(1, 4)
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Rules of the game

- The game is played only once by two players
- The players choose simultaneously their actions (no communication)
- Each player receives his payoff depending of all the chosen actions
- The goal of each player is to maximise his own payoff

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Hypotheses made in strategic games

- The players are **intelligent** (*i.e. they reason perfectly and quickly*)
- The players are **rational** (*i.e. they want to maximise their payoff*)
- The players are **selfish** (*i.e. they only care for their own payoff*)

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(S, S) is the only rational choice!

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Strategic games

Definition

A *strategic game* G is a triple $(N, (A_i)_{i \in N}, (P_i)_{i \in N})$ where:

- N is the **finite** and **non empty** set of players,
- A_i is the **non empty** set of actions of player i ,
- $P_i : A_1 \times \cdots \times A_N \rightarrow \mathbb{R}$ is the **payoff function** of player i .

	C	S
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Nash equilibrium

Nash Equilibrium - Definition

Let (N, A_i, P_i) be a strategic game and $a = (a_i)_{i \in N}$ be a *strategy profile*.

We say that $a = (a_i)_{i \in N}$ is a *Nash equilibrium* iff

$$\forall i \in N \quad \forall b_i \in A_i \quad P_i(b_i, a_{-i}) \leq P_i(a_i, a_{-i})$$

	C	S
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(S,S) is the unique Nash equilibrium

Do all the finite matrix games have a Nash equilibrium?

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No: matching pennies

	L	R
L	$(1, -1)$	$(-1, 1)$
R	$(-1, 1)$	$(1, -1)$

Nash equilibria in mixed strategies

	L	R
L	(1, -1)	(-1, 1)
R	(-1, 1)	(1, -1)

The following profile is a *Nash equilibrium in mixed strategies*:

$$\sigma_1 = \begin{cases} \text{L} & \text{with proba } \frac{1}{2} \\ \text{R} & \text{with proba } \frac{1}{2} \end{cases} \quad \text{and} \quad \sigma_2 = \begin{cases} \text{L} & \text{with proba } \frac{1}{2} \\ \text{R} & \text{with proba } \frac{1}{2} \end{cases}$$

whose *expected payoff* is 0.

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whose *expected payoff* is 0.

Nash Theorem [1950]

Every finite game admits mixed Nash equilibria.

Symmetric games

	X	Y
X	(α, α)	(γ, δ)
Y	(δ, γ)	(β, β)

Symmetric games

A symmetric game is a game $(N, (A_i)_{i \in N}, (P_i)_{i \in N})$ where:

- $A_1 = A_2 = \dots = A_N$
- $\forall (a_1, \dots, a_N) \in A_1 \times \dots \times A_N, \forall \pi$ permutations, $\forall k$, we have that $P_{\pi(k)}(a_1, \dots, a_N) = P_k(a_{\pi(1)}, \dots, a_{\pi(k)})$

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- Special case of 2-players: $\forall (a_1, a_2) \in A_1 \times A_2, P_2(a_1, a_2) = P_1(a_2, a_1)$

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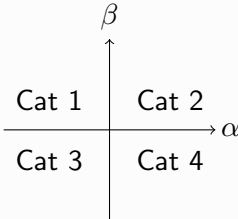
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Symmetric Nash Equilibrium

A Nash equilibrium $(\sigma_1, \dots, \sigma_N)$ is said symmetric when $\sigma_1 = \dots = \sigma_N$.

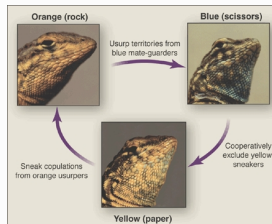
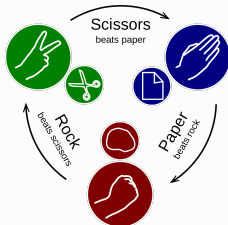
Example 1: 2×2 games - The 4 categories

	X	Y
X	(α, α)	$(0, 0)$
Y	$(0, 0)$	(β, β)



- Cat 1: $\alpha < 0$ et $\beta > 0$. NE= $\{(Y, Y)\}$
- Cat 2: $\alpha, \beta > 0$. NE= $\{(X, X), (Y, Y), (\sigma, \sigma)\}$ with $\sigma = \left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$
- Cat 3: $\alpha, \beta < 0$. NE= $\{(X, Y), (Y, X), (\sigma, \sigma)\}$ with $\sigma = \left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$
- Cat 4: $\alpha > 0$ et $\beta < 0$. NE= $\{(X, X)\}$

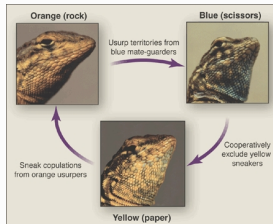
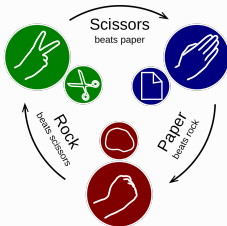
Example 2: The generalised Rock-Scissors-Paper Games



	R	S	P
R	$(1, 1)$	$(2 + a, 0)$	$(0, 2 + a)$
S	$(0, 2 + a)$	$(1, 1)$	$(2 + a, 0)$
P	$(2 + a, 0)$	$(0, 2 + a)$	$(1, 1)$

(The original RPS game is obtained when $a = 0$)

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P	(2 + a, 0)	(0, 2 + a)	(1, 1)

(The original RPS game is obtained when $a = 0$)

A unique Nash equilibrium (σ, σ, σ) , where $\sigma = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Some results on symmetric games

Theorem [Cheng et al, 2004]

Every 2-strategy symmetric game (i.e. $|A_i| = 2$) admits a (pure) Nash equilibrium. *But it might not be symmetric...*

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- no longer true if not “symmetric”: Matching pennies

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	L	R
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- not necessarily symmetric: anti-coordination game

	X	Y
X	(0, 0)	(1, 1)
Y	(1, 1)	(0, 0)

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The point of view of evolutionary games

	C	S
C	(3, 3)	(1, 4)
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Rules of the game

- We have a **large** population of individuals
- Individuals are repeatedly drawn at random to play the above game
- The payoffs are supposed to represent the gain in biological fitness or reproductive value

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Hypotheses made in evolutionary games

- Each individual is **genetically programmed** to play either C or S
- The individuals are no more **intelligent**, nor **rational**, nor **selfish**

Can an existing population resist to the invasion of a mutant?

The point of view of evolutionary games

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The strategy **S** is evolutionary stable, facing an invasion of the mutant strategy **C**.

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Evolutionary Stable Strategy: robustness to mutations

Evolutionary Stable Strategy

We say that σ is an **evolutionary stable strategy (ESS)** if

- (σ, σ) is a Nash equilibrium
- $\forall \sigma' (\neq \sigma) \quad P(\sigma', \sigma) = P(\sigma, \sigma) \implies P(\sigma', \sigma') < P(\sigma', \sigma)$

Thus if (σ, σ) is a **strict** Nash equilibrium, then σ is an ESS.

	A	B		C	S
A	(1, 1)	(1, 1)	C	(1, 1)	(1, 1)
B	(1, 1)	(2, 2)	S	(1, 1)	(0, 0)

- (A,A), (B,B) and (C,C) are Nash equilibria.
- A is not an **ESS**.
- B and C are **ESS**.

Evolutionary Stable Strategy - Alternative definition

- Imagine a population composed of a unique species σ
- A small proportion ϵ of the population mutates to a new species σ'
- The new population is thus $\epsilon\sigma' + (1 - \epsilon)\sigma$

Proposition

A strategy σ is an **ESS** iff $\forall \sigma' (\neq \sigma) \exists \epsilon_0 \in (0, 1) \forall \epsilon \in (0, \epsilon_0)$

$$P(\sigma, \epsilon\sigma' + (1 - \epsilon)\sigma) > P(\sigma', \epsilon\sigma' + (1 - \epsilon)\sigma)$$

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$$P(\sigma, \epsilon\sigma' + (1 - \epsilon)\sigma) > P(\sigma', \epsilon\sigma' + (1 - \epsilon)\sigma)$$

Static concept: it suffices to study the one-shot game

Evolutionary Stable Strategy - 2×2 games

			β	
			\uparrow	
	X	Y		
X	(α, α)	$(0, 0)$	Cat 1	Cat 2
Y	$(0, 0)$	(β, β)	Cat 3	Cat 4
				$\rightarrow \alpha$

Cat 1 : NE = $\{(Y, Y)\}$

ESS = $\{Y\}$

Cat 2 : NE = $\{(X, X), (Y, Y), (\sigma, \sigma)\}$

ESS = $\{X, Y\}$

Cat 3 : NE = $\{(X, Y), (Y, X), (\sigma, \sigma)\}$

ESS = $\{\sigma\}$

Cat 4 : NE = $\{(X, X)\}$

ESS = $\{X\}$

The evolution of a population - intuitively

Population composed of several species

Variation of popu. the species = Popu. of the species \times Advantage of the species

Advantage of the species = Fitness of the species $-$ Average fitness of all species

The evolution of a population - more formally (1)

- We consider a population where individuals are divided into n species. Individuals of species i are programmed to play the pure strategy a_i .
- We denote by $p_i(t)$ the number of individuals of species i at time t .
- The **total population at time t** is given by

$$p(t) = p_1(t) + \cdots + p_n(t)$$

- The **population state at time t** is given by

$$\sigma(t) = (\sigma_1(t), \dots, \sigma_n(t)) \quad \text{where} \quad \sigma_i(t) = \frac{p_i(t)}{p(t)}$$

The evolution of a population - more formally (2)

The evolution of the state of the population is given by:

The replicator dynamics (RD)

$$\frac{d}{dt}\sigma_i(t) = (P(a_i, \sigma(t)) - P(\sigma(t), \sigma(t))) \cdot \sigma_i(t)$$

Theorem

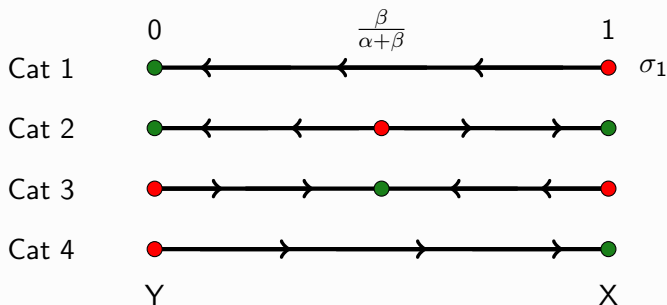
Given any initial condition $\sigma(0) \in \Delta(A)$, the above system of differential equations always admits a unique solution.

The replicator dynamics - 2×2 games

	X	Y	Cat 1	Cat 2	$\begin{cases} \frac{d}{dt}\sigma_1(t) = (\alpha\sigma_1(t) - \beta\sigma_2(t)) \cdot \sigma_1(t)\sigma_2(t) \\ \frac{d}{dt}\sigma_2(t) = (\beta\sigma_2(t) - \alpha\sigma_1(t)) \cdot \sigma_1(t)\sigma_2(t) \end{cases}$
X	(α, α)	$(0, 0)$			
Y	$(0, 0)$	(β, β)	Cat 3	Cat 4	

$\Delta(A) = \{(\sigma_1, \sigma_2) \in [0, 1]^2 \mid \sigma_1 + \sigma_2 = 1\} \simeq [0, 1]$, where σ_1 is the proportion of X

The solutions $(\sigma_1(t), 1 - \sigma_1(t))$ of the (RD) behave as follows:



Results

There are several results relating various notions of “static” stability:

- Nash equilibrium,
- Evolutionary Stable Strategy,
- Neutrally Stable Strategy...

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with various notions of “dynamic” stability:

- stationary points,
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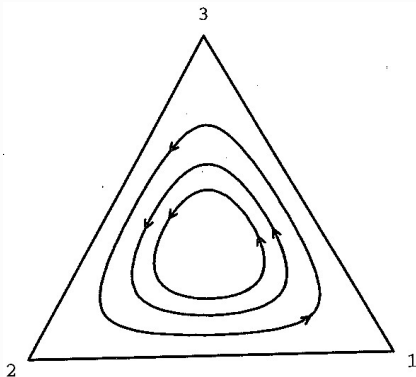
Theorems

- If $\sigma \in \Delta$ is Lyapunov stable, then σ is a NE.
- If $\sigma \in \Delta$ is an ESS, then σ is asymptotically stable.

Rock-Scissors-Paper

$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is Lyapunov stable but not asymptotically stable.

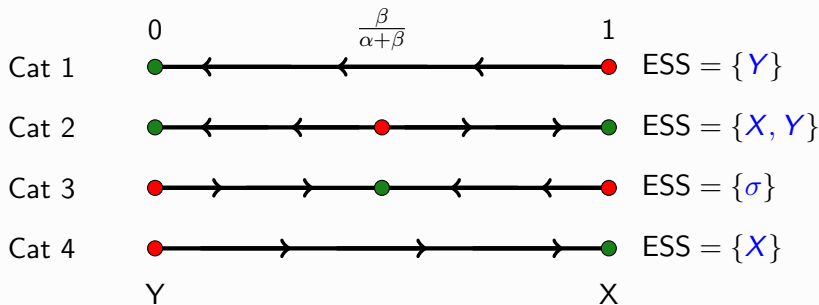
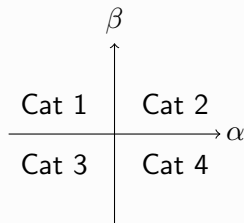
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The picture is taken from *Evolutionary game theory* by J.W. Weibull.

2 × 2 games - RD Vs ESS

	X	Y
X	(α, α)	$(0, 0)$
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● Asymptotically stable

● Stationary

An alternative dynamics

Replicator dynamics

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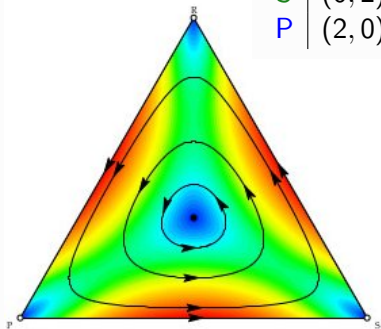
Alternative hypothesis: offspring react **smartly** to the mixture of past strategies played by the opponents, by playing a **best-reply strategy** to this mixture

Best-reply dynamics

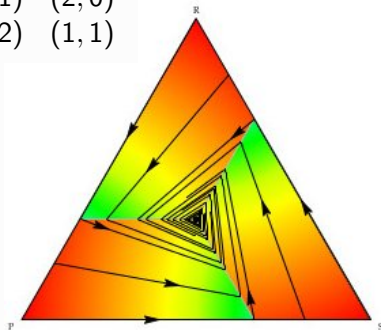
Variation of Strategy Mixture = Best-Reply Strategy – Current Strategy Mixture

Replicator vs Best-reply

	R	S	P
R	(1, 1)	(2, 0)	(0, 2)
S	(0, 2)	(1, 1)	(2, 0)
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Replicator dynamics



Best-reply dynamics

Pictures taken from *Evolutionary game theory* by W. H. Sandholm

Other dynamics

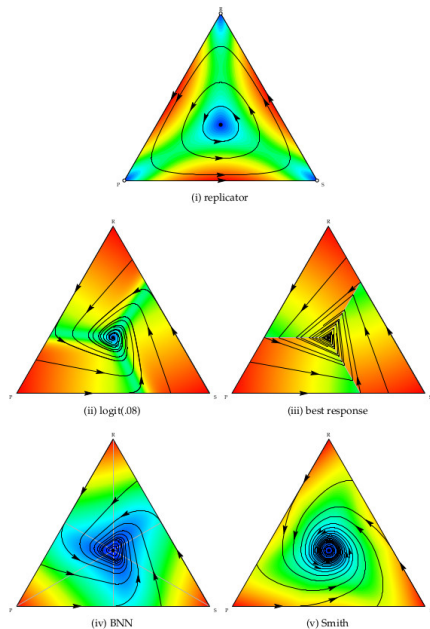


Figure 1: Five basic deterministic dynamics in standard Rock-Paper-Scissors. Colors represent speeds: red

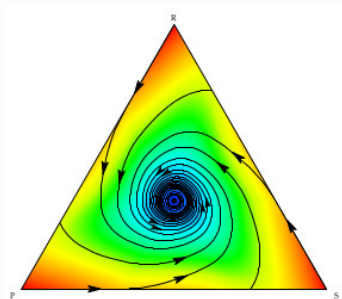
Static vs dynamic approach

Static approach

Equilibria

Dynamic approach

Stable Points



Picture taken from *Evolutionary game theory* by W. H. Sandholm

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- Find immediately a good strategy is concretely impossible

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- With enough different plays, will we eventually stabilize?

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Static vs dynamic approach

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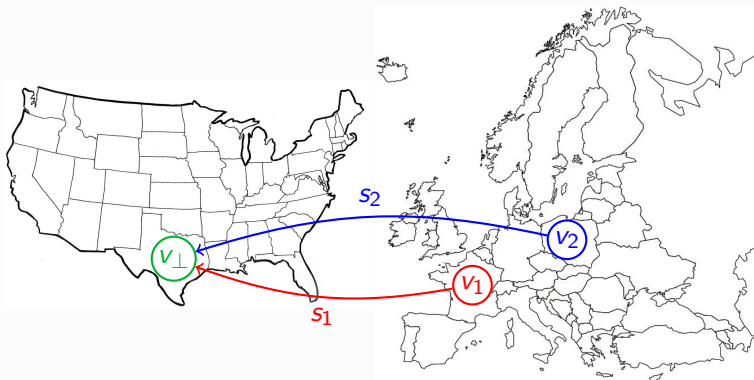
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Our Goal

- Apply this idea of improvement/mutation on games played on graphs
- Prove stabilisation via reduction/minor of games
- Show some links with interdomain routing

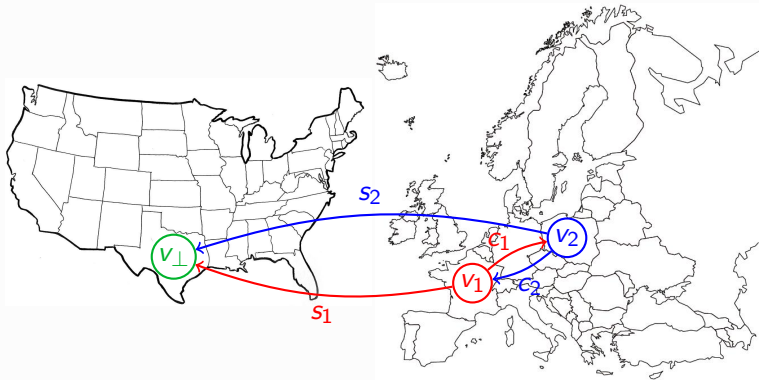
Interdomain routing problem

Two service providers: v_1 and v_2 want to route packets to v_{\perp} .



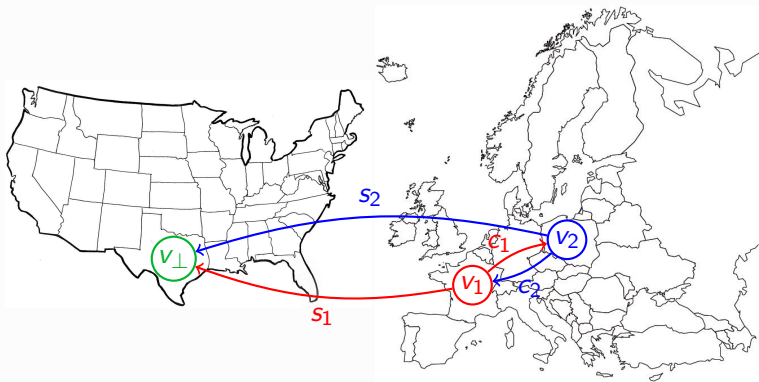
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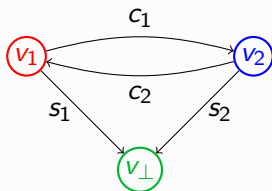


v_1 prefers the route $v_1 v_2 v_\perp$ to the route $v_1 v_\perp$ (preferred to $(v_1 v_2)^\omega$)

v_2 prefers the route $v_2 v_1 v_\perp$ to the route $v_2 v_\perp$ (preferred to $(v_2 v_1)^\omega$)

Interdomain routing problem as a game played on a graph

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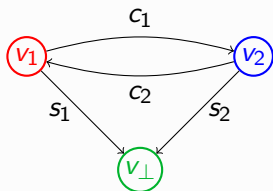


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$$v_1 v_\perp \prec_1 v_1 v_2 v_\perp \quad \text{and} \quad v_2 v_\perp \prec_2 v_2 v_1 v_\perp$$

Games played on a graph – The strategic game approach

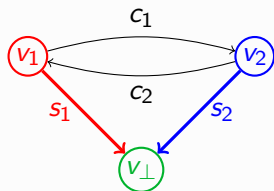


	c_2	s_2
c_1	(0, 0)	(2, 1)
s_1	(1, 2)	(1, 1)

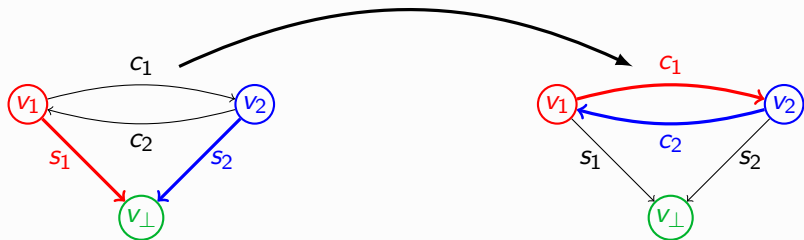
2 Nash equilibria: (c_1, s_2) and (s_1, c_2)

Static vision of the game: players are perfectly informed and supposed to be **intelligent**, **rational** and **selfish**

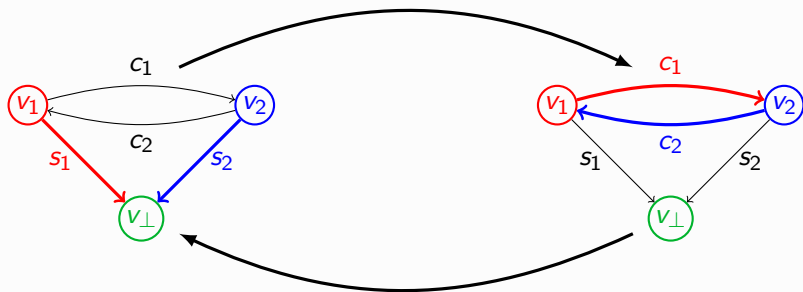
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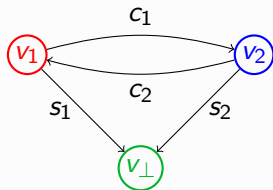


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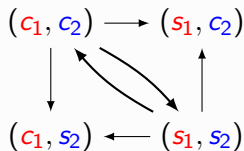


Asynchronous nature of the network could block the packets in an undesirable cycle...

Interdomain routing problem - open problem



The game G



The graph of the dynamics: $G\langle\rightarrow\rangle$

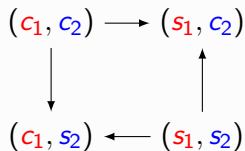
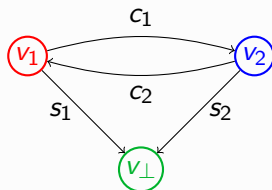
Identify necessary and sufficient conditions on G such that $G\langle\rightarrow\rangle$ has no cycle

Ideally, the conditions should be algorithmically simple, locally testable...

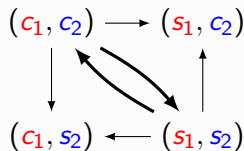
Numerous interesting partial solutions proposed in the literature

Games played on a graph – The evolutionary approach

Different dynamics



D_1 with no cycle



D_2 with a cycle

Outline

- 1 A brief review of strategic games
 - Nash equilibrium
 - Symmetric two-player games
- 2 Evolutionary game theory
 - Evolutionary Stable Strategy
 - The Replicator Dynamics
 - Other Dynamics
- 3 Games played on graphs
 - Two examples of dynamics
 - Relations that maintain termination
 - More realistic conditions
 - Application to interdomain routing

Positional 1-step dynamics $\xrightarrow{P1}$

$$\text{profile}_1 \xrightarrow{P1} \text{profile}_2$$

if:

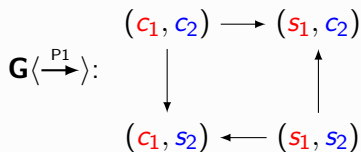
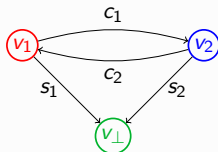
- a single player changes **at a single node**
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Positional Concurrent Dynamics $\xrightarrow{\text{PC}}$

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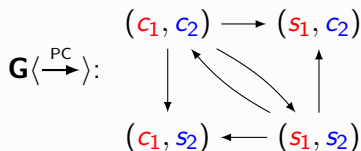
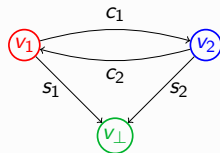
- one or several players change **at a single node**
- all players that change **intend** to improve their outcome
- but synchronous changes may result in worst outcomes...

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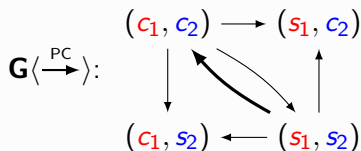
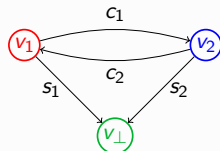


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both players **intend** to reach their best outcome ($v_1 v_\perp \prec_1 v_1 v_2 v_\perp$ and $v_2 v_\perp \prec_2 v_2 v_1 v_\perp$), even if they do not manage to do it (as the reached outcome is $(v_1 v_2)^\omega$ and $(v_2 v_1)^\omega$)

Questions

What condition should \mathbf{G} satisfy to ensure that

$\mathbf{G}\langle\rightarrow\rangle$ has no cycles, i.e. dynamics \rightarrow terminates on \mathbf{G} ?

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What should \mathbf{G}_1 and \mathbf{G}_2 have in common to ensure that

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Simulation relation on dynamics graphs

G simulates G' ($G' \sqsubseteq G$) if **all that G' can do, G can do it too.**

$$\forall \text{profile}'_1 \longrightarrow \forall \text{profile}'_2$$

$$\sqcap \mid$$
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$$\forall \text{profile}_1$$

Simulation relation on dynamics graphs

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 $\sqcap \parallel$ $\sqcap \parallel$

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$$\begin{array}{ccc} \forall & & \forall \\ \text{profile}'_1 & \longrightarrow & \text{profile}'_2 \\ \sqcap \parallel & & \sqcap \parallel \\ \forall & & \exists \\ \text{profile}_1 & \longrightarrow & \text{profile}_2 \end{array}$$

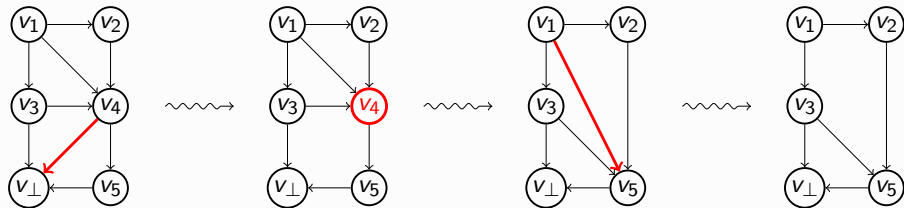
Folklore

If $\mathbf{G}_1 \langle \rightarrow_1 \rangle$ simulates $\mathbf{G}_2 \langle \rightarrow_2 \rangle$ and the dynamics \rightarrow_1 terminates on \mathbf{G}_1 , then the dynamics \rightarrow_2 terminates on \mathbf{G}_2 .

Relation between games

\mathbf{G}' is a minor of \mathbf{G} if it is obtained by a succession of operations:

- deletion of an edge (and all the corresponding outcomes);
- deletion of an isolated node;
- deletion of a node v with a single edge $v \rightarrow v'$ and no predecessor $u \rightarrow v$ such that $u \rightarrow v'$.



Relation between simulation and minor

Theorem

If \mathbf{G}' is a minor of \mathbf{G} , then $\mathbf{G} \langle \xrightarrow{P1} \rangle$ simulates $\mathbf{G}' \langle \xrightarrow{P1} \rangle$. In particular, if $\xrightarrow{P1}$ terminates for \mathbf{G} , it terminates for \mathbf{G}' too.

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Remark: $\mathbf{G}\langle\overset{\text{P1}}{\rightarrow}\rangle \sqsubseteq \mathbf{G}\langle\overset{\text{PC}}{\rightarrow}\rangle$

More realistic conditions

Adding fairness

- Termination might be too strong to ask in interdomain routing...
- Every router that wants to change its decision will have the opportunity to do it in the future...
- Study of *fair termination*

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More realistic dynamics

Consider *best reply* variants $\xrightarrow{\text{bP1}}$ and $\xrightarrow{\text{bPC}}$ of the two dynamics, where each player that modifies its strategy changes in the best possible way

What results?

Previous theorem

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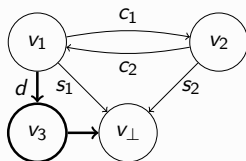
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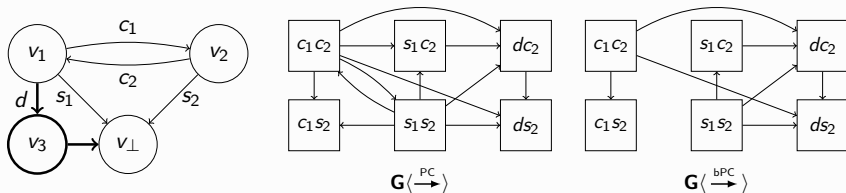


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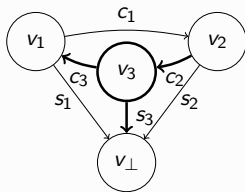
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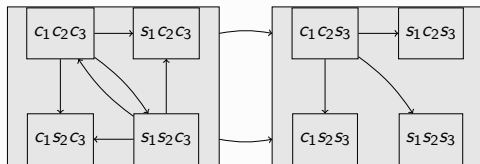
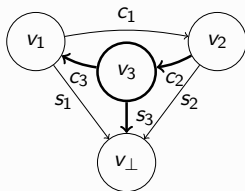


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Theorem

If \mathbf{G}' is a *dominant minor* of \mathbf{G} , then $\xrightarrow{\text{bPC}} / \xrightarrow{\text{bP1}}$ fairly terminates for \mathbf{G} if and only if it fairly terminates for \mathbf{G}' .

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- Use of simulations that are partially invertible...

Interdomain routing

- Particular case of game with one target for all players (reachability game) and players owning a single node (router)

Theorem [Sami, Shapira, Zohar, 2009]

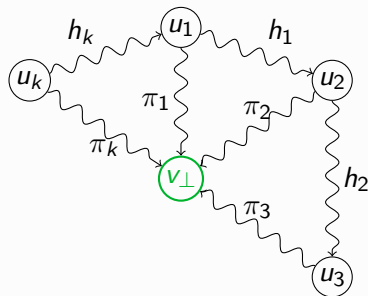
If \mathbf{G} is a one-target game for which $\xrightarrow{\text{bPC}}$ fairly terminates, then it has exactly one *equilibrium*.

Interdomain routing

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Theorem [Griffin, Shepherd, Wilfong, 2002]

There exists a pattern, called *dispute wheel* such that if \mathbf{G} is a one-target game that has no dispute wheels, then $\xrightarrow{\text{bPC}}$ fairly terminates.



$$\forall 1 \leq i \leq k \quad \pi_i \prec_{u_i} h_i \pi_{i+1}$$

...

Reciprocal?

Theorem

There exists a stronger pattern, called *strong dispute wheel*, such that if $\xrightarrow{\text{PC}}$ terminates for \mathbf{G} , then \mathbf{G} has no strong dispute wheel.

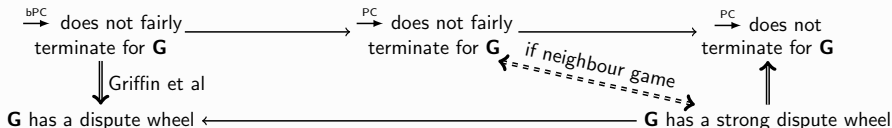
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If \mathbf{G} satisfies a locality condition on the preferences, then \xrightarrow{PC} fairly terminates for \mathbf{G} if and only if \mathbf{G} has no strong dispute wheel.



Reciprocal?

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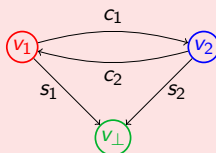
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Theorem

Finding a strong dispute wheel in \mathbf{G} can be tested by searching whether \mathbf{G} contains the following game as a minor:



Summary

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- Different possible dynamics
- Conditions for (fair) termination
- Use of game minors and graph simulations
- In the article, non-positional strategies are also considered

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Thank you! Questions?