Dynamics on Games: Simulation-Based Techniques and Applications to Routing

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ANR Delta
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Slides partly borrowed from Thomas Brihaye and Marion Hallet
Work published at FSTTCS 2019
Outline

1. A brief review of strategic games
   - Nash equilibrium
   - Symmetric two-player games

2. Evolutionary game theory
   - Evolutionary Stable Strategy
   - The Replicator Dynamics
   - Other Dynamics

3. Games played on graphs
   - Two examples of dynamics
   - Relations that maintain termination
   - More realistic conditions
   - Application to interdomain routing
Prisoner’s dilemma

<table>
<thead>
<tr>
<th>Prisoner A</th>
<th>Prisoner B</th>
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<tbody>
<tr>
<td><strong>Confess</strong></td>
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Equivalently (since only the relative order of payoffs matters): 

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The point of view of strategic games

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Rules of the game

- The game is played only once by two players
- The players choose simultaneously their actions (no communication)
- Each player receives his payoff depending of all the chosen actions
- The goal of each player is to maximise his own payoff
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Hypotheses made in strategic games

- The players are **intelligent** (*i.e.* they reason perfectly and quickly)
- The players are **rational** (*i.e.* they want to maximise their payoff)
- The players are **selfish** (*i.e.* they only care for their own payoff)
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(S,S) is the only rational choice!

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Strategic games

Definition

A strategic game $G$ is a triple $(N, (A_i)_{i \in N}, (P_i)_{i \in N})$ where:

- $N$ is the finite and non empty set of players,
- $A_i$ is the non empty set of actions of player $i$,
- $P_i : A_1 \times \cdots \times A_N \rightarrow \mathbb{R}$ is the payoff function of player $i$.

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Nash equilibrium

Nash Equilibrium - Definition

Let \((N, A_i, P_i)\) be a strategic game and \(a = (a_i)_{i \in N}\) be a strategy profile. We say that \(a = (a_i)_{i \in N}\) is a Nash equilibrium iff

\[
\forall i \in N \ \forall b_i \in A_i \quad P_i(b_i, a_{-i}) \leq P_i(a_i, a_{-i})
\]

\[
\begin{array}{c|cc}
 & C & S \\
\hline
C & (3, 3) & (1, 4) \\
S & (4, 1) & (2, 2)
\end{array}
\]

\((S,S)\) is the unique Nash equilibrium
Do all the finite matrix games have a Nash equilibrium?

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Do all the finite matrix games have a Nash equilibrium?

No: matching pennies

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Nash equilibria in mixed strategies

\[
\begin{array}{c|cc}
 & L & R \\
\hline
L & (1, -1) & (-1, 1) \\
R & (-1, 1) & (1, -1) \\
\end{array}
\]

The following profile is a Nash equilibrium in mixed strategies:

\[
\sigma_1 = \begin{cases} 
L & \text{with proba } \frac{1}{2} \\
R & \text{with proba } \frac{1}{2}
\end{cases}
\quad \text{and} \quad
\sigma_2 = \begin{cases} 
L & \text{with proba } \frac{1}{2} \\
R & \text{with proba } \frac{1}{2}
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\]

whose expected payoff is 0.
Nash equilibria in mixed strategies

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**Nash Theorem [1950]**

Every finite game admits mixed Nash equilibria.
Symmetric games

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A symmetric game is a game \((N, (A_i)_{i \in N}, (P_i)_{i \in N})\) where:

- \(A_1 = A_2 = \cdots = A_N\)
- \(\forall (a_1, \ldots, a_N) \in A_1 \times \cdots \times A_N, \forall \pi \text{ permutations}, \forall k,\) we have that 
  \[ P_{\pi(k)}(a_1, \ldots, a_N) = P_k(a_{\pi(1)}, \ldots, a_{\pi(k)}) \]
Symmetric games

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- Special case of 2-players: \(\forall (a_1, a_2) \in A_1 \times A_2, P_2(a_1, a_2) = P_1(a_2, a_1)\)
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A symmetric game is a game \( \left( N, (A_i)_{i \in N}, (P_i)_{i \in N} \right) \) where:

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Symmetric Nash Equilibrium

A Nash equilibrium \( (\sigma_1, \ldots, \sigma_N) \) is said symmetric when \( \sigma_1 = \cdots = \sigma_N \).
Example 1: $2 \times 2$ games - The 4 categories

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<tr>
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<td>$(0, 0)$</td>
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- **Cat 1**: $\alpha < 0$ et $\beta > 0$. NE=$\{(Y, Y)\}$
- **Cat 2**: $\alpha, \beta > 0$. NE=$\{(X, X), (Y, Y), (\sigma, \sigma)\}$ with $\sigma = \left( \frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta} \right)$
- **Cat 3**: $\alpha, \beta < 0$. NE=$\{(X, Y), (Y, X), (\sigma, \sigma)\}$ with $\sigma = \left( \frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta} \right)$
- **Cat 4**: $\alpha > 0$ et $\beta < 0$. NE=$\{(X, X)\}$
Example 2: The generalised Rock-Scissors-Paper Games

Uta stansburiana - The side-blotched lizard

The populations for these lizards cycle on a six year basis. When he read that lizards of the species Uta stansburiana were essentially engaged in a game with rock-paper-scissors structure John Maynard Smith exclaimed: They have read my book!

\[
\begin{array}{cccc}
 & R & S & P \\
 R & (1, 1) & (2 + a, 0) & (0, 2 + a) \\
 S & (0, 2 + a) & (1, 1) & (2 + a, 0) \\
 P & (2 + a, 0) & (0, 2 + a) & (1, 1) \\
\end{array}
\]

(The original RPS game is obtained when \( a = 0 \))
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P & (2 + a, 0) & (0, 2 + a) & (1, 1) \\
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\]

(The original RPS game is obtained when \( a = 0 \))

A unique Nash equilibrium \((\sigma, \sigma, \sigma)\), where \( \sigma = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \).
Some results on symmetric games

**Theorem [Cheng et al, 2004]**

Every 2-strategy symmetric game (i.e. $|A_i| = 2$) admits a (pure) Nash equilibrium. *But it might not be symmetric...*
Some results on symmetric games

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- no longer true if not “2-strategy”: RPS...
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- no longer true if not “symmetric”: Matching pennies

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- no longer true if not “2-strategy”: RPS...
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```
\begin{array}{c|cc}
 & L & R \\
\hline
L & (1,-1) & (-1,1) \\
R & (-1,1) & (1,-1) \\
\end{array}
```

- not necessarily symmetric: anti-coordination game

```
\begin{array}{c|cc}
 & X & Y \\
\hline
X & (0,0) & (1,1) \\
Y & (1,1) & (0,0) \\
\end{array}
```
Outline

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Rules of the game

- We have a **large** population of individuals
- Individuals are repeatedly drawn at random to play the above game
- The payoffs are supposed to represent the gain in biological fitness or reproductive value
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Hypotheses made in evolutionary games

- Each individual is **genetically programmed** to play either C or S
- The individuals are no more **intelligent**, nor **rational**, nor **selfish**

Can an existing population resist to the invasion of a mutant?
The point of view of evolutionary games

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The strategy S is evolutionary stable, facing an invasion of the mutant strategy C.

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Evolutionary Stable Strategy: robustness to mutations

**Evolutionary Stable Strategy**

We say that $\sigma$ is an **evolutionary stable strategy (ESS)** if

- $(\sigma, \sigma)$ is a Nash equilibrium
- $\forall \sigma' \neq \sigma \ P(\sigma', \sigma) = P(\sigma, \sigma) \implies P(\sigma', \sigma') < P(\sigma', \sigma)$

Thus if $(\sigma, \sigma)$ is a **strict** Nash equilibrium, then $\sigma$ is an ESS.

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- (A,A), (B,B) and (C,C) are Nash equilibria.
- A is not an ESS.
- B and C are ESS.
Evolutionary Stable Strategy - Alternative definition

- Imagine a population composed of a unique species $\sigma$
- A small proportion $\epsilon$ of the population mutes to a new species $\sigma'$
- The new population is thus $\epsilon\sigma' + (1-\epsilon)\sigma$

**Proposition**

A strategy $\sigma$ is an **ESS** iff $\forall \sigma' (\neq \sigma) \exists \epsilon_0 \in (0, 1) \forall \epsilon \in (0, \epsilon_0)$

$$P(\sigma, \epsilon\sigma' + (1-\epsilon)\sigma) > P(\sigma', \epsilon\sigma' + (1-\epsilon)\sigma)$$
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$$P(\sigma, \epsilon\sigma' + (1 - \epsilon)\sigma) > P(\sigma', \epsilon\sigma' + (1 - \epsilon)\sigma)$$

Static concept: it suffices to study the one-shot game
Evolutionary Stable Strategy - $2 \times 2$ games

\[
\begin{array}{c|cc}
 & X & Y \\
\hline
X & (\alpha, \alpha) & (0, 0) \\
Y & (0, 0) & (\beta, \beta) \\
\end{array}
\]

Cat 1: NE = \{(Y, Y)\} \quad \text{ESS} = \{Y\}

Cat 2: NE = \{(X, X), (Y, Y), (\sigma, \sigma)\} \quad \text{ESS} = \{X, Y\}

Cat 3: NE = \{(X, Y), (Y, X), (\sigma, \sigma)\} \quad \text{ESS} = \{\sigma\}

Cat 4: NE = \{(X, X)\} \quad \text{ESS} = \{X\}
The evolution of a population - intuitively

Population composed of several species

\[ \text{Variation of popu. the species} = \text{Popu. of the species} \times \text{Advantage of the species} \]

\[ \text{Advantage of the species} = \text{Fitness of the species} - \text{Average fitness of all species} \]
We consider a population where individuals are divided into $n$ species. Individuals of species $i$ are programmed to play the pure strategy $a_i$.

We denote by $p_i(t)$ the number of individuals of species $i$ at time $t$.

The **total population at time** $t$ is given by

$$p(t) = p_1(t) + \cdots + p_n(t)$$

The **population state at time** $t$ is given by

$$\sigma(t) = (\sigma_1(t), \ldots, \sigma_n(t)) \quad \text{where} \quad \sigma_i(t) = \frac{p_i(t)}{p(t)}$$
The evolution of a population - more formally (2)

The evolution of the state of the population is given by:

**The replicator dynamics (RD)**

\[
\frac{d}{dt} \sigma_i(t) = (P(a_i, \sigma(t)) - P(\sigma(t), \sigma(t))) \cdot \sigma_i(t)
\]

**Theorem**

Given any initial condition \( \sigma(0) \in \Delta(A) \), the above system of differential equations always admits a unique solution.
The replicator dynamics - 2 × 2 games

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Cat 1  Cat 2

\[\begin{align*}
\frac{d}{dt} \sigma_1(t) &= (\alpha \sigma_1(t) - \beta \sigma_2(t)) \cdot \sigma_1(t) \sigma_2(t) \\
\frac{d}{dt} \sigma_2(t) &= (\beta \sigma_2(t) - \alpha \sigma_1(t)) \cdot \sigma_1(t) \sigma_2(t)
\end{align*}\]

\[\Delta(A) = \{(\sigma_1, \sigma_2) \in [0, 1]^2 \mid \sigma_1 + \sigma_2 = 1\} \simeq [0, 1], \text{ where } \sigma_1 \text{ is the proportion of } X\]

The solutions \((\sigma_1(t), 1 - \sigma_1(t))\) of the (RD) behave as follows:

\[
\begin{array}{c}
\text{Cat 1} \\
0 & \frac{\beta}{\alpha + \beta} & 1 \\
\text{Cat 2} \\
\text{Cat 3} \\
\text{Cat 4}
\end{array}\]
Results

There are several results relating various notions of “static” stability:
- Nash equilibrium,
- Evolutionary Stable Strategy,
- Neutrally Stable Strategy...
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There are several results relating various notions of “static” stability:

- Nash equilibrium,
- Evolutionary Stable Strategy,
- Neutrally Stable Strategy...

with various notions of “dynamic” stability:

- stationary points,
- Lyapunov stable points,
- asymptotically stable point...
Results

There are several results relating various notions of “static” stability:
- Nash equilibrium,
- Evolutionary Stable Strategy,
- Neutrally Stable Strategy...

with various notions of “dynamic” stability:
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- asymptotically stable point ...

Theorems
- If $\sigma \in \Delta$ is Lyapunov stable, then $\sigma$ is a NE.
- If $\sigma \in \Delta$ is an ESS, then $\sigma$ is asymptotically stable.
Rock-Scissors-Paper

\( \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \) is Lyapunov stable but not asymptotically stable.

<table>
<thead>
<tr>
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<th>R</th>
<th>S</th>
<th>P</th>
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<tbody>
<tr>
<td>R</td>
<td>(1,1)</td>
<td>(2,0)</td>
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</tbody>
</table>

The picture is taken from *Evolutionnary game theory* by J.W. Weibull.
2 × 2 games - RD Vs ESS

\[
\begin{array}{c|cc}
 & X & Y \\
\hline
X & (\alpha, \alpha) & (0, 0) \\
Y & (0, 0) & (\beta, \beta) \\
\end{array}
\]

\[
\begin{align*}
\text{Cat 1} & \quad \text{Cat 2} \\
\text{Cat 3} & \quad \text{Cat 4}
\end{align*}
\]

\[
\begin{align*}
\beta & \quad \alpha \\
\text{ESS} = \{Y\} & \\
\text{ESS} = \{X, Y\} & \\
\text{ESS} = \{\sigma\} & \\
\text{ESS} = \{X\} & \\
\end{align*}
\]

- Asymptotically stable
- Stationary
An alternative dynamics

### Replicator dynamics

<table>
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<tr>
<th>Variation of popu. the species</th>
<th>Popu. of the species $\times$ Advantage of the species</th>
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<tr>
<td>Advantage of the species</td>
<td>Fitness of the species $-$ Average fitness of all species</td>
</tr>
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</table>
An alternative dynamics

**Replicator dynamics**

Variation of popu. the species = Popu. of the species \( \times \) Advantage of the species

Advantage of the species = Fitness of the species — Average fitness of all species

Alternative hypothesis: offspring react **smartly** to the mixture of past strategies played by the opponents, by playing a **best-reply strategy** to this mixture

**Best-reply dynamics**

Variation of Strategy Mixture = Best-Reply Strategy — Current Strategy Mixture
Replicator vs Best-reply

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Replicator dynamics

Best-reply dynamics

Pictures taken from *Evolutionnary game theory* by W. H. Sandholm
Other dynamics

Figure 1: Five basic deterministic dynamics in standard Rock-Paper-Scissors. Colors represent speeds: red
If we discover a new game, finding immediately a good strategy is concretely impossible. If we play several times, we will improve our strategy. With enough different plays, will we eventually stabilize? If so, will this strategy be a good strategy?

Our goal is to apply this idea of improvement/mutation on games played on graphs and prove stabilization via reduction/minor of games. Show some links with interdomain routing.
Static vs dynamic approach

Static approach  Dynamic approach

Equilibria  Stable Points

If we discover a new game

- Find immediately a good strategy is concretely impossible
Static vs dynamic approach

**Static approach**

Equilibria

**Dynamic approach**

Stable Points

If we discover a new game

- Find immediately a good strategy is concretely impossible
- If we play several times, we will improve our strategy
Static vs dynamic approach

**Static approach**
- Equilibria

**Dynamic approach**
- Stable Points

---

If we discover a new game
- Find immediately a good strategy is concretely impossible
- If we play several times, we will improve our strategy
- With enough different plays, will we eventually stabilize?
Static vs dynamic approach

**Static approach**

Equilibria

**Dynamic approach**

Stable Points

---

If we discover a new game

- Find immediately a good strategy is concretely impossible
- If we play several times, we will improve our strategy
- With enough different plays, will we eventually stabilize?
- If so, will this strategy be a *good* strategy?
Static vs dynamic approach

**Static approach**

- Equilibria

**Dynamic approach**

- Stable Points

---

**If we discover a new game**

- Find immediately a good strategy is concretely impossible
- If we play several times, we will improve our strategy
- With enough different plays, will we eventually stabilize?
- If so, will this strategy be a *good* strategy?

**Our Goal**

- Apply this idea of improvement/mutation on games played on graphs
- Prove stabilisation via reduction/minor of games
- Show some links with interdomain routing
Interdomain routing problem

Two service providers: $v_1$ and $v_2$ want to route packets to $v_\perp$. 

Diagram showing the routing paths from $v_1$ and $v_2$ to $v_\perp$. The labels $s_1$ and $s_2$ indicate the connections between the service providers and the target domain.
Interdomain routing problem

Two service providers: $v_1$ and $v_2$ want to route packets to $v_{\perp}$. 
Interdomain routing problem

Two service providers: $v_1$ and $v_2$ want to route packets to $v_\perp$.

$v_1$ prefers the route $v_1v_2v_\perp$ to the route $v_1v_\perp$ (preferred to $(v_1v_2)^\omega$).

$v_2$ prefers the route $v_2v_1v_\perp$ to the route $v_2v_\perp$ (preferred to $(v_2v_1)^\omega$).
Interdomain routing problem as a game played on a graph

Two service providers: $v_1$ and $v_2$ want to route packets to $v_\perp$.

$v_1$ prefers the route $v_1v_2v_\perp$ to the route $v_1v_\perp$ (preferred to $(v_1v_2)^\omega$)

$v_2$ prefers the route $v_2v_1v_\perp$ to the route $v_2v_\perp$ (preferred to $(v_2v_1)^\omega$)

$v_1v_\perp \prec_1 v_1v_2v_\perp$ and $v_2v_\perp \prec_2 v_2v_1v_\perp$
Games played on a graph – The strategic game approach

2 Nash equilibria: \((c_1, s_2)\) and \((s_1, c_2)\)

**Static** vision of the game: players are perfectly informed and supposed to be **intelligent**, **rational** and **selfish**
Games played on a graph – The evolutionary approach
Games played on a graph – The evolutionary approach
Asynchronous nature of the network could block the packets in an undesirable cycle...
Interdomain routing problem - open problem

The game $G(c_1, c_2) ightarrow (s_1, c_2)$

The graph of the dynamics: $G\langle \rightarrow \rangle$

Identify necessary and sufficient conditions on $G$ such that $G\langle \rightarrow \rangle$ has no cycle

Ideally, the conditions should be algorithmically simple, locally testable...

Numerous interesting partial solutions proposed in the literature

Daggitt, Gurney, Griffin. Asynchronous convergence of policy-rich distributed Bellman-Ford routing protocols. 2018
Games played on a graph – The evolutionary approach

Different dynamics

\(D_1\) with no cycle

\(D_2\) with a cycle
Outline

1. A brief review of strategic games
   - Nash equilibrium
   - Symmetric two-player games

2. Evolutionary game theory
   - Evolutionary Stable Strategy
   - The Replicator Dynamics
   - Other Dynamics

3. Games played on graphs
   - Two examples of dynamics
   - Relations that maintain termination
   - More realistic conditions
   - Application to interdomain routing
Positional 1-step dynamics $\xrightarrow{P1}$

$\text{profile}_1 \xrightarrow{P1} \text{profile}_2$

if:

- a single player changes at a single node
- this player improves his own outcome
Positional 1-step dynamics $^{P1}$

profile$_1 \xrightarrow{^{P1}}$ profile$_2$

if:

- a single player changes at a single node
- this player improves his own outcome

Diagram:

$G^{^{P1}}:$

$$(c_1, c_2) \rightarrow (s_1, c_2)$$

$$(c_1, s_2) \leftarrow (s_1, s_2)$$
Positional Concurrent Dynamics $\rightarrow$ 

$\text{profile}_1 \xrightarrow{\text{PC}} \text{profile}_2$

if

- one or several players change at a single node
- all players that change intend to improve their outcome
- but synchronous changes may result in worst outcomes...
Positional Concurrent Dynamics

\[ \text{profile}_1 \xrightarrow{\text{PC}} \text{profile}_2 \]

if

- one or several players change **at a single node**
- all players that change **intend** to improve their outcome
- but synchronous changes may result in worst outcomes...

\[ G^{\text{PC}} \]:

\[
\begin{align*}
(c_1, c_2) &\xrightarrow{\text{PC}} (s_1, c_2) \\
(c_1, s_2) &\xleftarrow{\text{PC}} (s_1, s_2)
\end{align*}
\]
Positional Concurrent Dynamics \( \rightarrow^{\text{PC}} \)

\[
\text{profile}_1 \rightarrow^{\text{PC}} \text{profile}_2
\]

if

- one or several players change \textbf{at a single node}
- all players that change \textbf{intend} to improve their outcome
- but synchronous changes may result in worst outcomes...

\[
\begin{align*}
\nu_1 &\leadsto c_1 \\
\nu_1 &\leadsto s_1 \\
\nu_\perp &\leadsto c_2 \\
\nu_\perp &\leadsto s_2 \\
\nu_2 &\leadsto c_1 \\
\nu_2 &\leadsto s_2
\end{align*}
\]

\[
G^{(\text{PC})}:
\begin{align*}
(c_1, c_2) &\rightarrow (s_1, c_2) \\
(c_1, s_2) &\rightarrow (s_1, s_2)
\end{align*}
\]

both players \textbf{intend} to reach their best outcome \((\nu_1 \nu_\perp \prec_1 \nu_1 \nu_2 \nu_\perp \text{ and } \nu_2 \nu_\perp \prec_2 \nu_2 \nu_1 \nu_\perp)\),
even if they do not manage to do it (as the reached outcome is \((\nu_1 \nu_2)^\omega \text{ and } (\nu_2 \nu_1)^\omega\))
Questions

What condition should $G$ satisfy to ensure that $G\langle\rightarrow\rangle$ has no cycles, i.e. dynamics $\rightarrow$ terminates on $G$?
Questions

What condition should $G$ satisfy to ensure that $G\langle\rightarrow\rangle$ has no cycles, i.e. dynamics $\rightarrow$ terminates on $G$?

What relations should $\rightarrow_1$ and $\rightarrow_2$ satisfy to ensure that $G\langle\rightarrow_1\rangle$ has no cycles if and only if $G\langle\rightarrow_2\rangle$ has no cycles?
Questions

What condition should $G$ satisfy to ensure that

$G\langle \rightarrow \rangle$ has no cycles, i.e. dynamics $\rightarrow$ terminates on $G$?

What relations should $\rightarrow_1$ and $\rightarrow_2$ satisfy to ensure that

$G\langle \rightarrow_1 \rangle$ has no cycles if and only if $G\langle \rightarrow_2 \rangle$ has no cycles?

What should $G_1$ and $G_2$ have in common to ensure that

$G_1\langle \rightarrow \rangle$ has no cycles if and only if $G_2\langle \rightarrow \rangle$ has no cycles?
Simulation relation on dynamics graphs

\[ G \text{ simulates } G' \ (G' \sqsubseteq G) \text{ if all that } G' \text{ can do, } G \text{ can do it too.} \]

\[ \forall \ \text{profile}'_1 \rightarrow \text{profile}'_2 \]

\[ \forall \ \text{profile}_1 \]

Folklore

If \( G_1 \langle 1 \rangle \) simulates \( G_2 \langle 2 \rangle \) and the dynamics \( 1 \) terminates on \( G_1 \), then the dynamics \( 2 \) terminates on \( G_2 \).
Simulation relation on dynamics graphs

$G$ simulates $G'$ ($G' \subseteq G$) if all that $G'$ can do, $G$ can do it too.

$$\forall \text{profile}'_1 \quad \forall \text{profile}'_2 \quad \text{profile}'_1 \rightarrow \text{profile}'_2$$

$$\nabla \quad \nabla$$

$$\forall \exists \quad \\forall \text{profile}_1 \quad \text{profile}_2$$
Simulation relation on dynamics graphs

$G$ simulates $G'$ ($G' \sqsubset G$) if all that $G'$ can do, $G$ can do it too.

\[
\forall \text{profile}'_{1} \quad \forall \text{profile}'_{2} \\
\text{profile}'_{1} \rightarrow \text{profile}'_{2}
\]

\[
\forall \text{profile}_{1} \quad \exists \text{profile}_{2} \\
\text{profile}_{1} \rightarrow \text{profile}_{2}
\]

Folklore

If $G_1\langle \rightarrow_1 \rangle$ simulates $G_2\langle \rightarrow_2 \rangle$ and the dynamics $\rightarrow_1$ terminates on $G_1$, then the dynamics $\rightarrow_2$ terminates on $G_2$. 
Relation between games

$G'$ is a minor of $G$ if it is obtained by a succession of operations:

- deletion of an edge (and all the corresponding outcomes);
- deletion of an isolated node;
- deletion of a node $v$ with a single edge $v \rightarrow v'$ and no predecessor $u \rightarrow v$ such that $u \rightarrow v'$.
Relation between simulation and minor

Theorem

If $G'$ is a minor of $G$, then $G\langle P_1 \rangle$ simulates $G'\langle P_1 \rangle$. In particular, if $P_1$ terminates for $G$, it terminates for $G'$ too.

Remark:

$G\langle P_1 \rangle \subseteq G\langle P_1 \rangle$
Relation between simulation and minor

**Theorem**

If \( G' \) is a minor of \( G \), then \( G\langle P_1 \rangle \) simulates \( G'\langle P_1 \rangle \). In particular, if \( P_1 \) terminates for \( G \), it terminates for \( G' \) too.

**Theorem**

If \( G' \) is a minor of \( G \), then \( G\langle P_C \rangle \) simulates \( G'\langle P_C \rangle \). In particular, if \( P_C \) terminates for \( G \), it terminates for \( G' \) too.

Remark: \( G\langle P_1 \rangle \sqsubseteq G\langle P_C \rangle \)
More realistic conditions

Adding fairness

- Termination might be too strong to ask in interdomain routing...
- Every router that wants to change its decision will have the opportunity to do it in the future...
- Study of *fair termination*
More realistic conditions

Adding fairness
- Termination might be too strong to ask in interdomain routing...
- Every router that wants to change its decision will have the opportunity to do it in the future...
- Study of fair termination

More realistic dynamics
Consider best reply variants $\text{bP}_1$ and $\text{bPC}$ of the two dynamics, where each player that modifies its strategy changes in the best possible way
What results?

**Previous theorem**

If $G'$ is a minor of $G$, then $G\langle \rightarrow_{PC} \rangle$ simulates $G'\langle \rightarrow_{PC} \rangle$. In particular, if $\rightarrow_{PC}$ terminates for $G$, it terminates for $G'$ too.

- Becomes false for best reply dynamics $\rightarrow_{bP1}$ and $\rightarrow_{bPC}$: the best reply dynamics could terminate in $G$ but not in the minor $G'$.
What results?

Previous theorem

If $G'$ is a minor of $G$, then $G^{\langle \rightarrow \rangle}$ simulates $G'^{\langle \rightarrow \rangle}$. In particular, if $\rightarrow$ terminates for $G$, it terminates for $G'$ too.

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What results?

Previous theorem

If $G'$ is a minor of $G$, then $G \langle \rightarrow^{PC} \rangle$ simulates $G' \langle \rightarrow^{PC} \rangle$. In particular, if $\rightarrow^{PC}$ terminates for $G$, it terminates for $G'$ too.

- Becomes false for best reply dynamics $\rightarrow^{bP_1}$ and $\rightarrow^{bPC}$: the best reply dynamics could terminate in $G$ but not in the minor $G'$
- Does not apply to fair termination: the dynamics could fairly terminate for $G$ (and not *terminate*) but not for $G'$
What results?

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If $G'$ is a minor of $G$, then $G \langle \rightarrow \rangle$ simulates $G' \langle \rightarrow \rangle$. In particular, if $\rightarrow$ terminates for $G$, it terminates for $G'$ too.

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![Diagram](image)
What results?

**Previous theorem**

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- The reciprocal does not hold...
What results?

Previous theorem

If $G'$ is a minor of $G$, then $G\langle PC \rangle$ simulates $G'\langle PC \rangle$. In particular, if $\longrightarrow$ terminates for $G$, it terminates for $G'$ too.

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- Does not apply to fair termination: the dynamics could fairly terminate for $G$ (and not terminate) but not for $G'$
- The reciprocal does not hold...

Theorem

If $G'$ is a dominant minor of $G$, then $bPC / bP1$ fairly terminates for $G$ if and only if it fairly terminates for $G'$. 
What results?

Previous theorem

If $G'$ is a minor of $G$, then $G\langle \rightarrow^{PC} \rangle$ simulates $G'\langle \rightarrow^{PC} \rangle$. In particular, if $\rightarrow^{PC}$ terminates for $G$, it terminates for $G'$ too.

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Theorem

If $G'$ is a dominant minor of $G$, then $\rightarrow^{bPC} / \rightarrow^{bP1}$ fairly terminates for $G$ if and only if it fairly terminates for $G'$.

- Use of simulations that are partially invertible...
Interdomain routing

- Particular case of game with one target for all players (reachability game) and players owning a single node (router)

Theorem [Sami, Shapira, Zohar, 2009]

If $G$ is a one-target game for which $\xrightarrow{bPC}$ fairly terminates, then it has exactly one equilibrium.
Interdomain routing

- Particular case of game with one target for all players (reachability game) and players owning a single node (router)

Theorem [Griffin, Shepherd, Wilfong, 2002]
There exists a pattern, called dispute wheel such that if $G$ is a one-target game that has no dispute wheels, then $bPC$ fairly terminates.

\[ \forall 1 \leq i \leq k \quad \pi_i \prec_{u_i} h_i \pi_{i+1} \]
Theorem

There exists a stronger pattern, called *strong dispute wheel*, such that if $\text{PC} \rightarrow$ terminates for $G$, then $G$ has no strong dispute wheel.
Theorem

There exists a stronger pattern, called *strong dispute wheel*, such that if \( \text{PC} \) terminates for \( G \), then \( G \) has no strong dispute wheel.

Theorem

If \( G \) satisfies a locality condition on the preferences, then \( \text{PC} \) fairly terminates for \( G \) if and only if \( G \) has no strong dispute wheel.

\[
\begin{align*}
\text{bPC} & \quad \text{does not fairly terminate for } G \\
\text{Griffin et al} & \quad \text{G has a dispute wheel} \\
\text{PC} & \quad \text{does not fairly terminate for } G \\
\text{PC} & \quad \text{does not terminate for } G \\
\text{PC} & \quad \text{has a strong dispute wheel}
\end{align*}
\]
Reciprocal?

**Theorem**

There exists a stronger pattern, called *strong dispute wheel*, such that if $\text{PC} \rightarrow$ terminates for $G$, then $G$ has no strong dispute wheel.

**Theorem**

If $G$ satisfies a locality condition on the preferences, then $\text{PC} \rightarrow$ fairly terminates for $G$ if and only if $G$ has no strong dispute wheel.

**Theorem**

Finding a strong dispute wheel in $G$ can be tested by searching whether $G$ contains the following game as a minor:

![Diagram of a graph with nodes $v_1$, $v_2$, and $v_⊥$, and edges $c_1$, $c_2$, $s_1$, and $s_2$.]
Summary

- Looking for equilibria in dynamics of $n$-player games
- Different possible dynamics
- Conditions for (fair) termination
- Use of game minors and graph simulations
- In the article, non-positional strategies are also considered
Summary

- Looking for equilibria in dynamics of $n$-player games
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Perspectives

- Still open to find a forbidden pattern/minor for fair termination of $\text{bPC} \rightarrow$ in one-target games
- Consider games with imperfect information: model of malicious router
- A better model of asynchronicity?
- Model fairness using probabilities?
Summary

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Thank you! Questions?