Weighted Automata vs Infinite-image Transducers: Specification and Decidability

Benjamin Monmege
LIF, Aix-Marseille Université, France

Meeting ANR DELTA, Porquerolles

Based on joint works with
Paul Gastin, Benedikt Bollig, Marc Zeitoun
as well as Théodore Lopez and Jean-Marc Talbot

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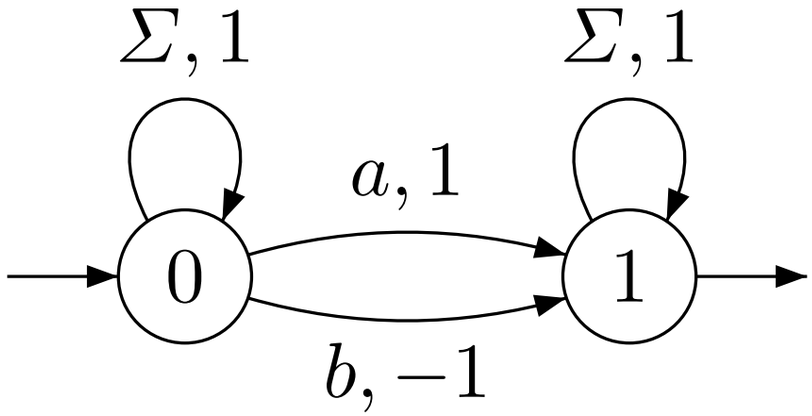
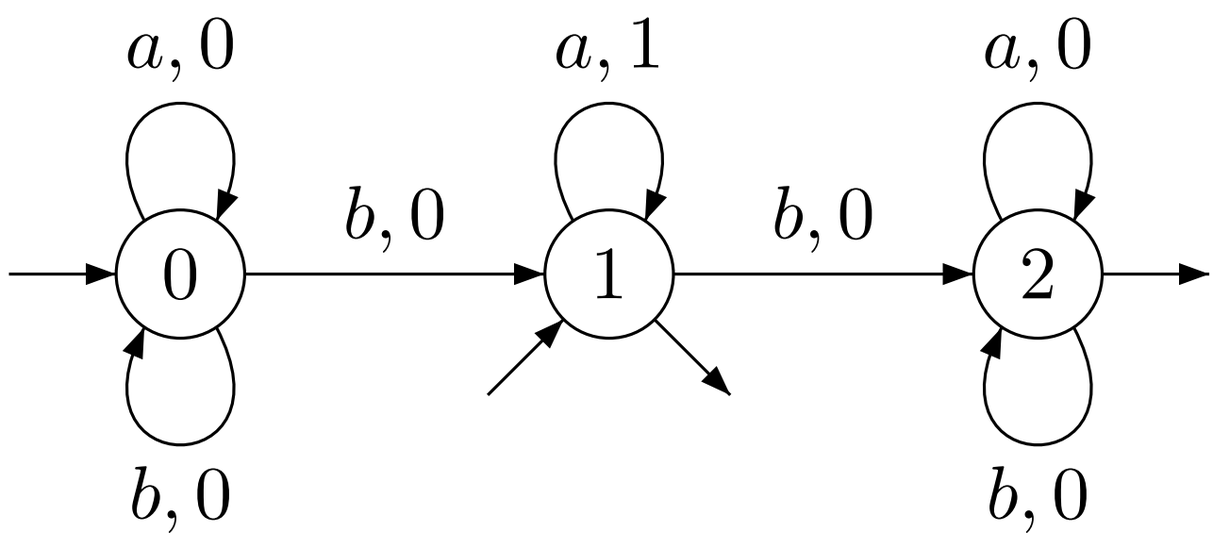
Weighted Automata





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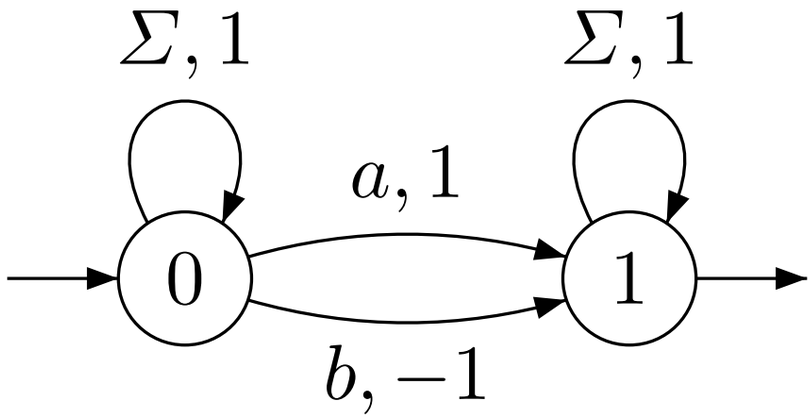
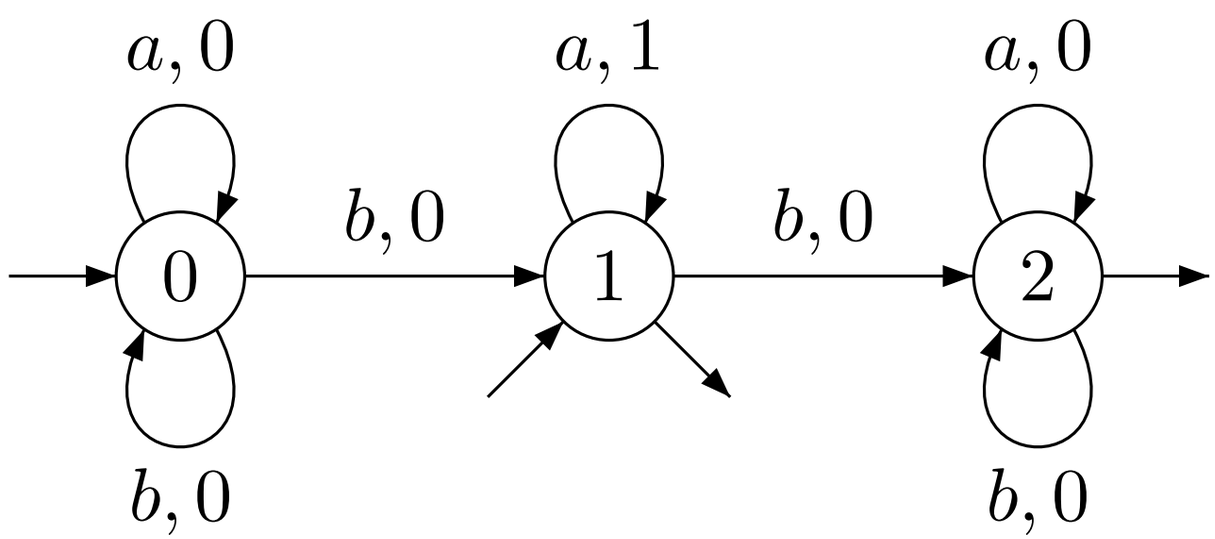
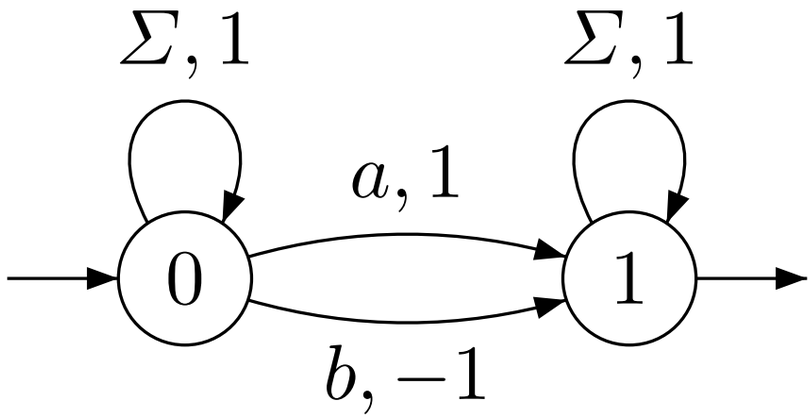
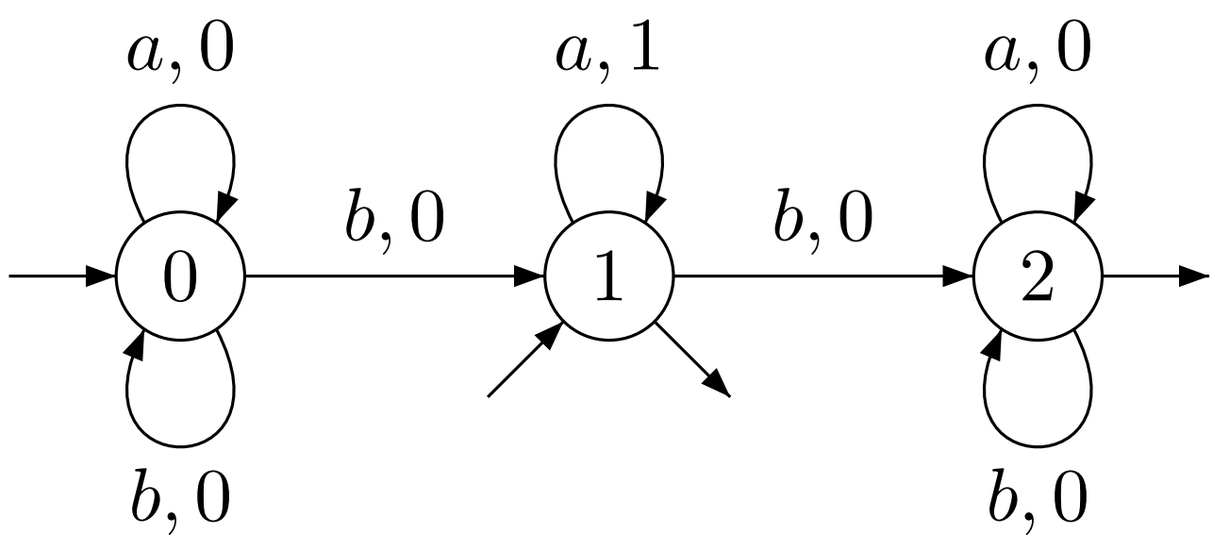




$(\mathbf{Z}, +, \times, 0, 1)$

Weighted Automata





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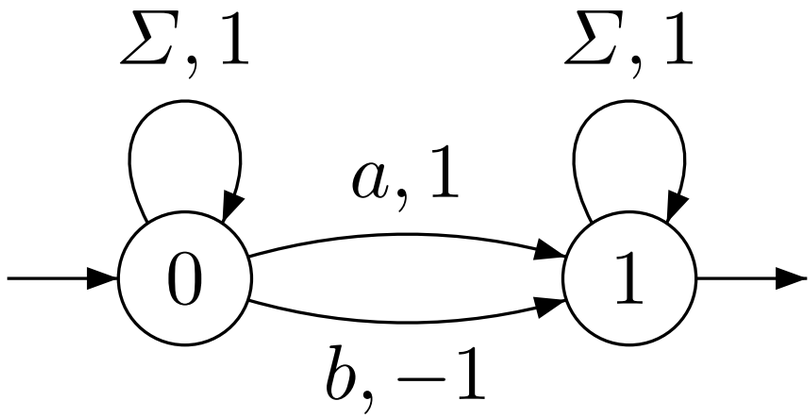
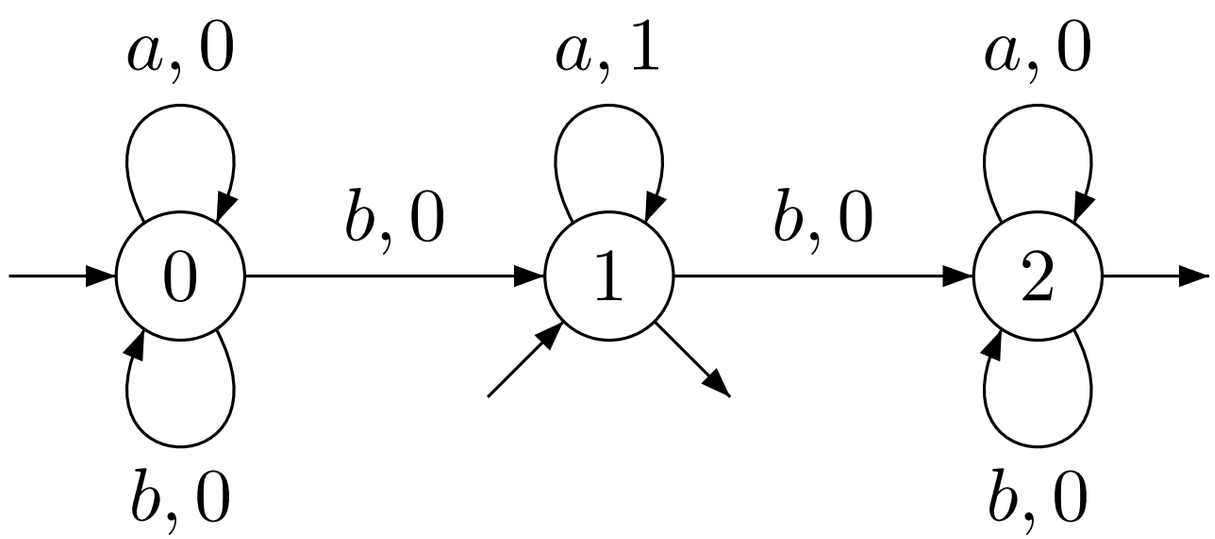
$$0 \xrightarrow{a,1} 1 \xrightarrow{b,1} 1 \xrightarrow{a,1} 1$$

1

Weighted Automata





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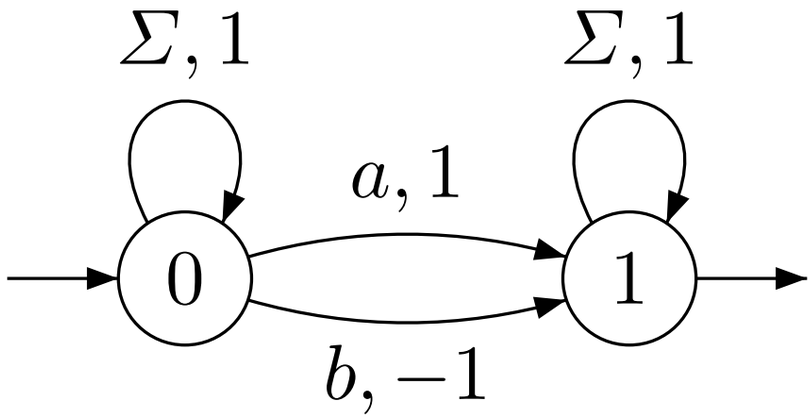

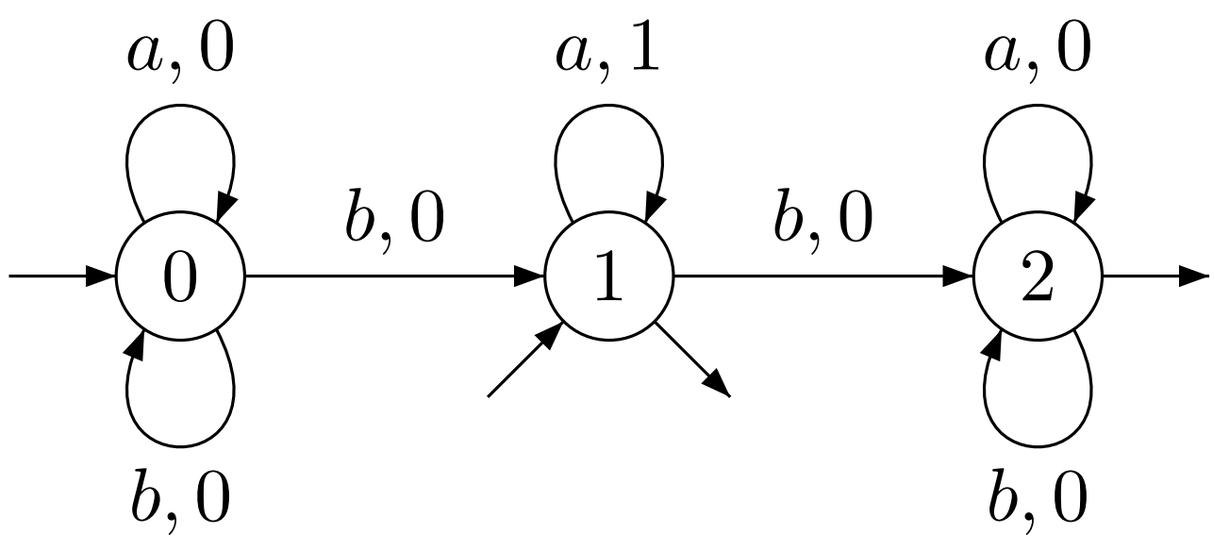

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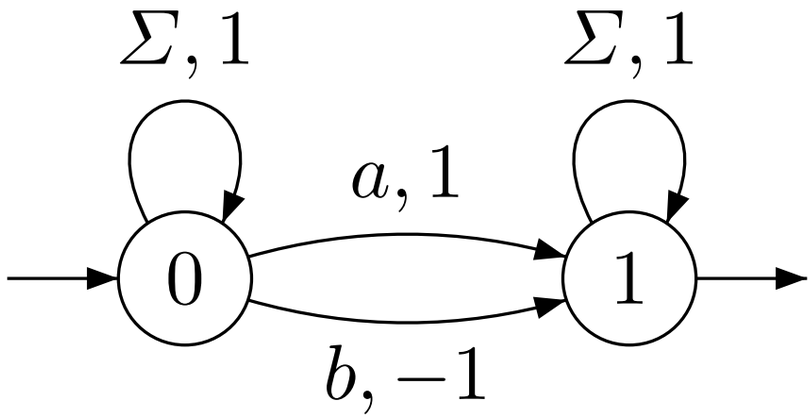

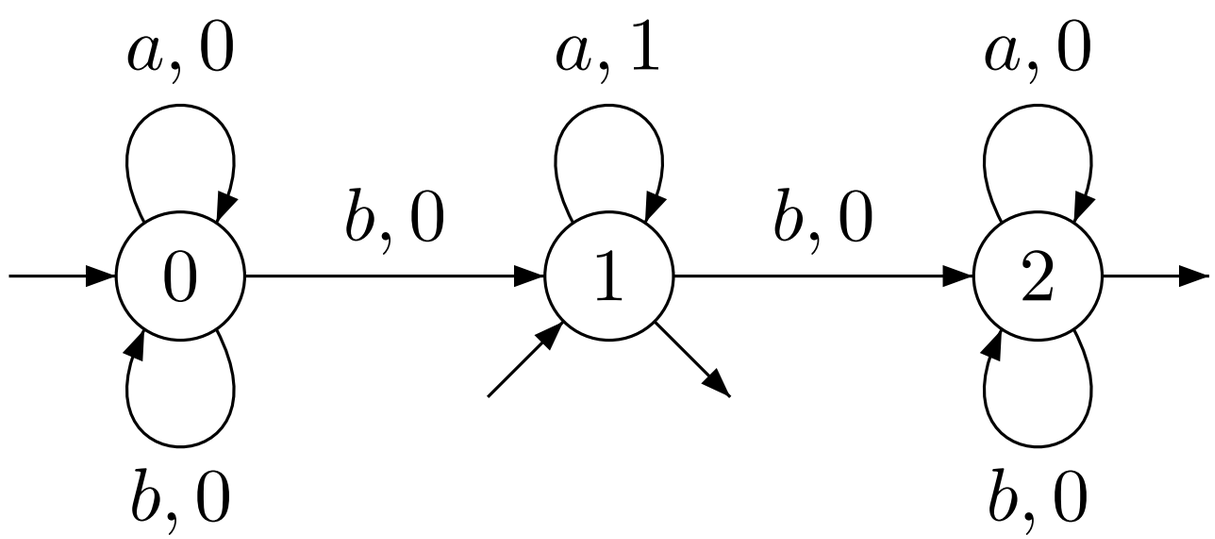

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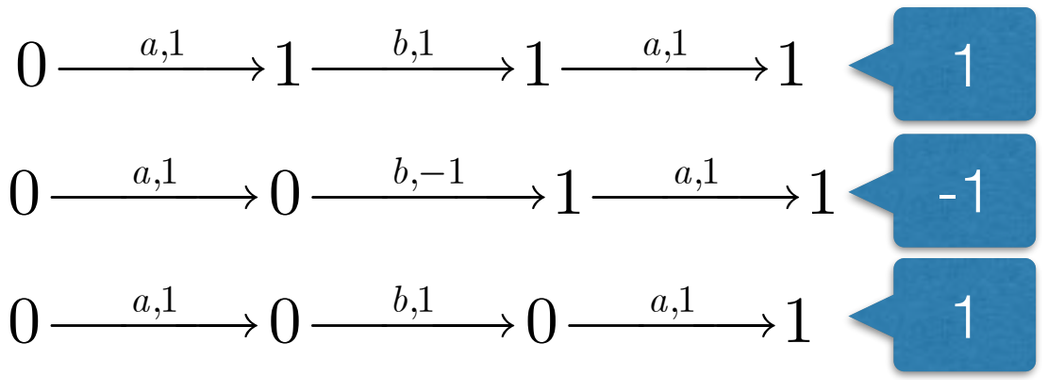
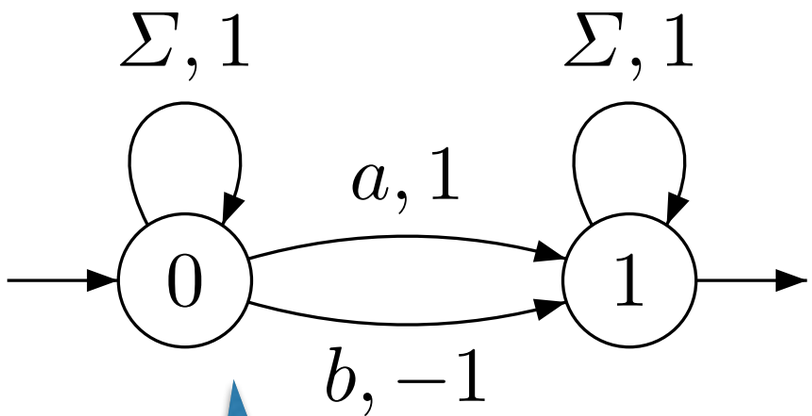


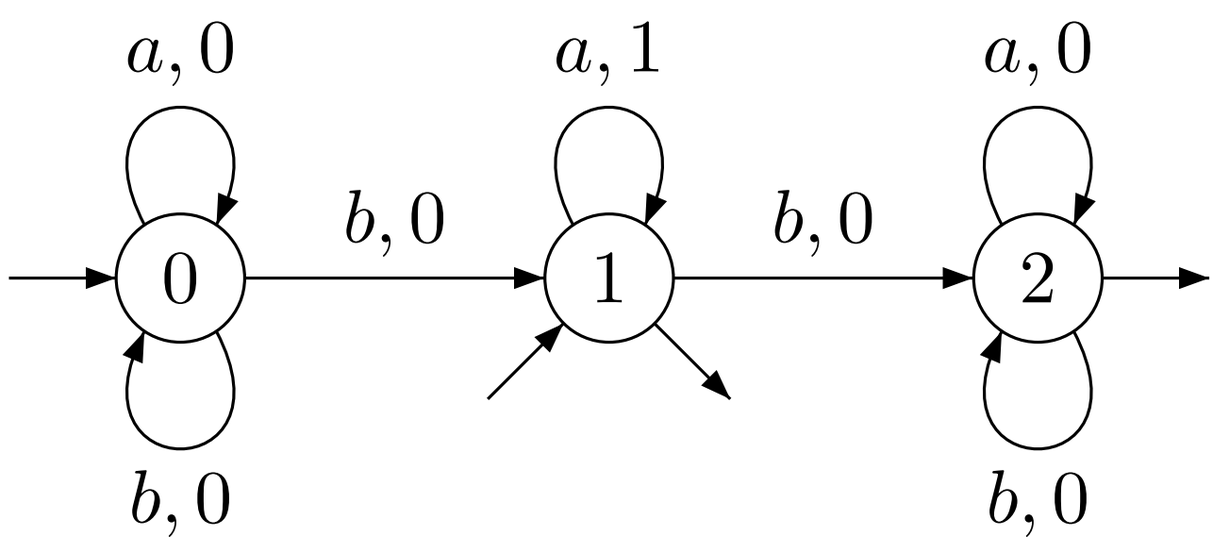
Semantics of *aba*: 1+(-1)+1 = 1

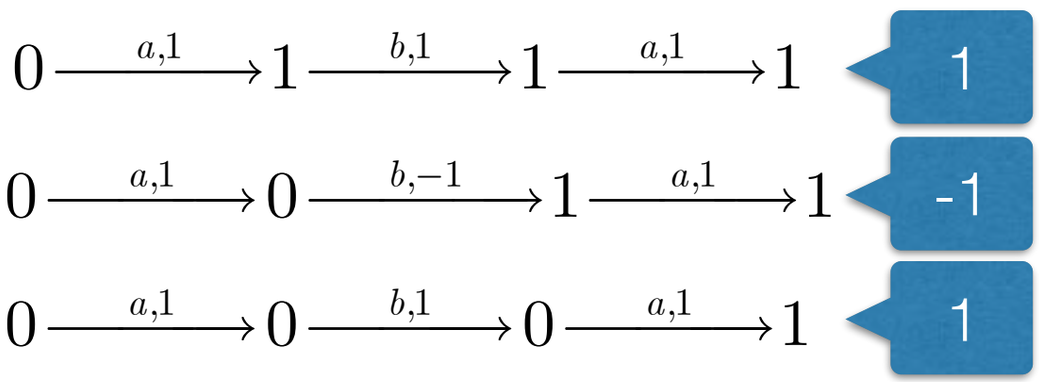
Weighted Automata



$\#_a(w)$ - $\#_b(w)$

$$(\mathbf{Z}, +, \times, 0, 1)$$





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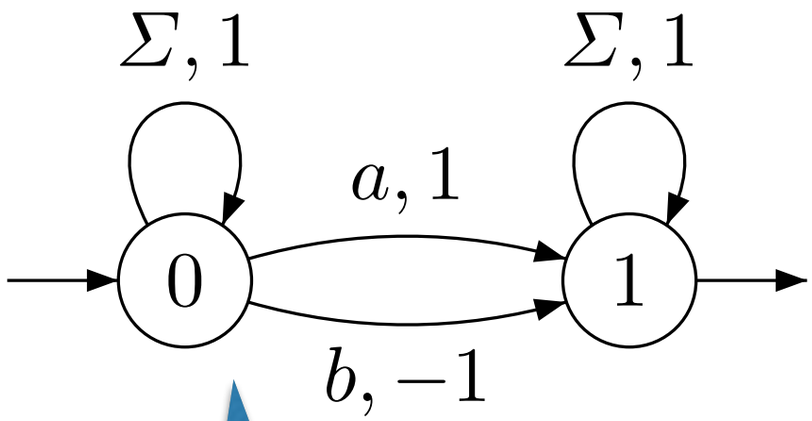

$\#_a(w) - \#_b(w)$

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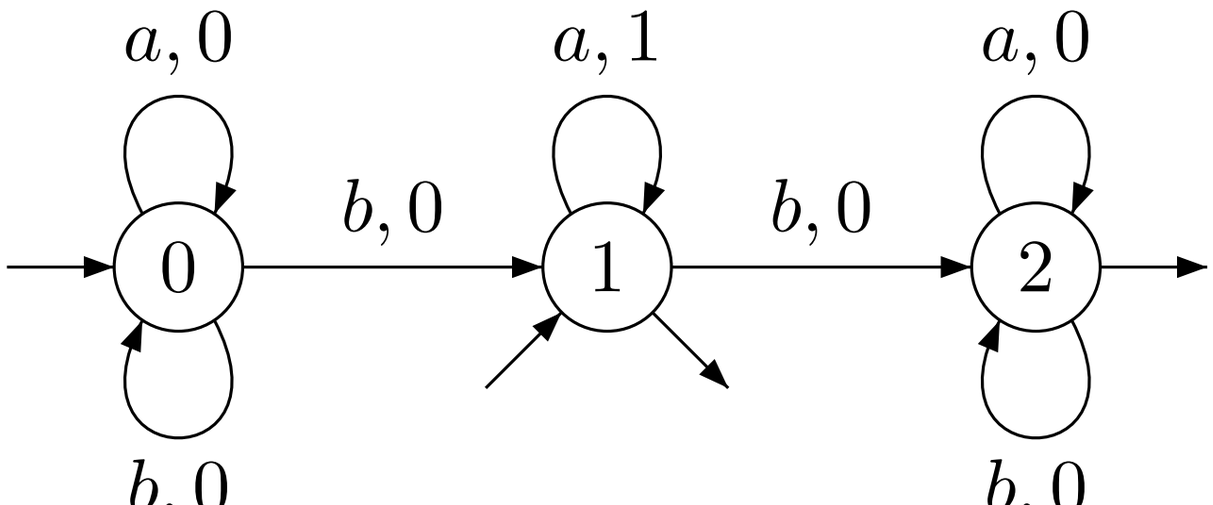

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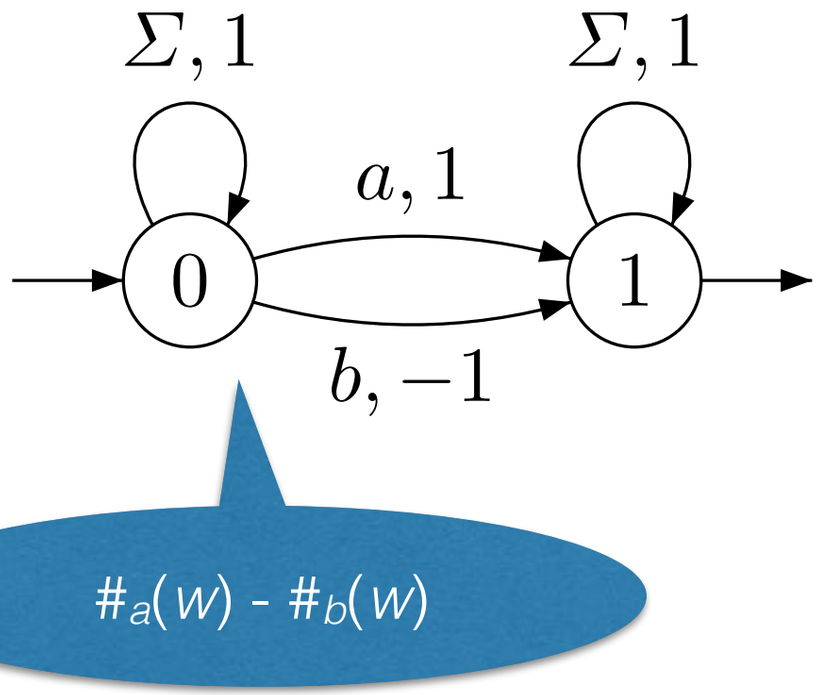

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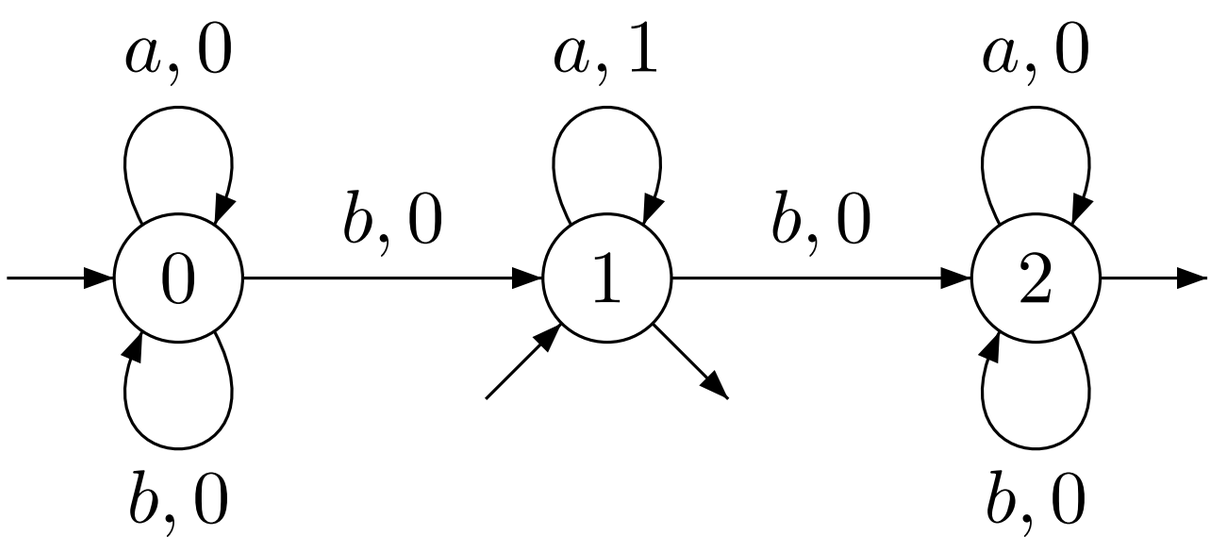

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$1 \xrightarrow{a,1} 1 \xrightarrow{a,1} 1 \xrightarrow{b,0} 2 \xrightarrow{a,0} 2$ 2

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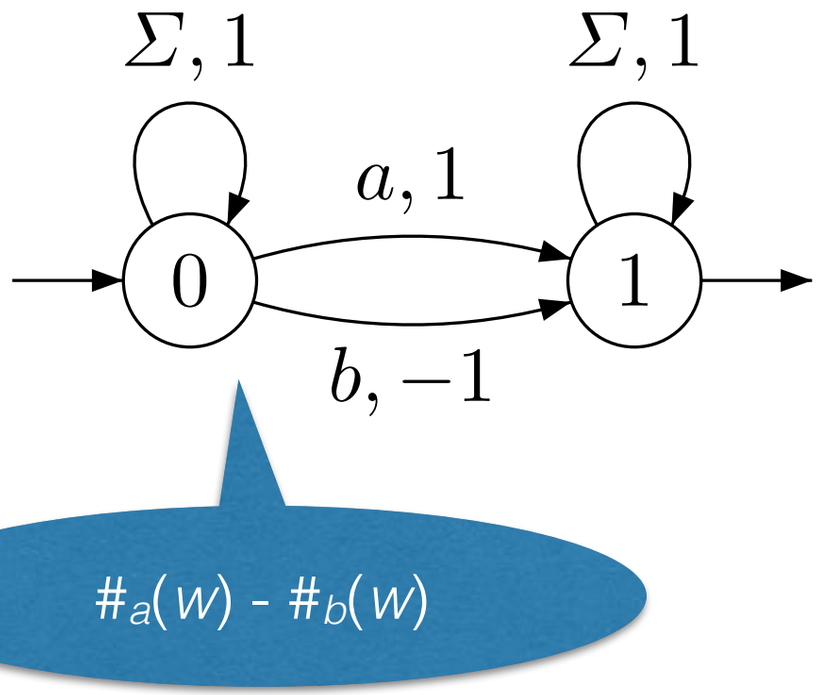
Weighted Automata



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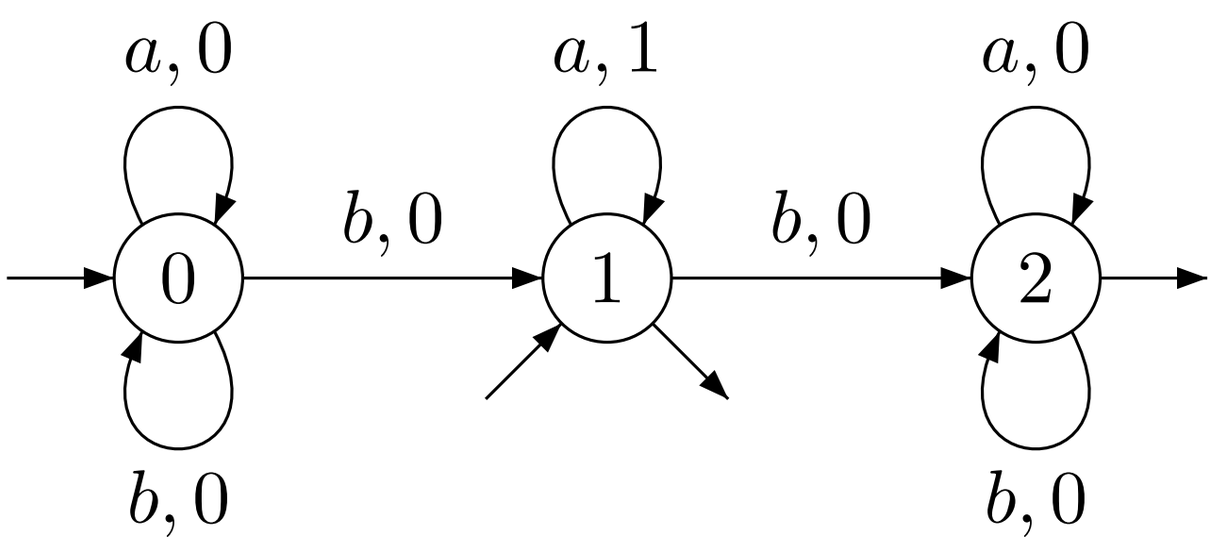

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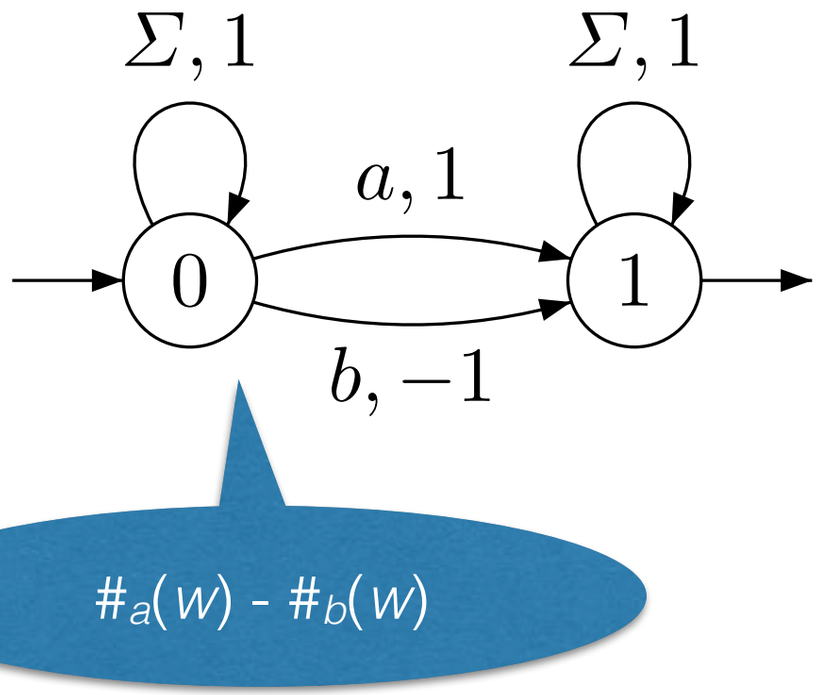

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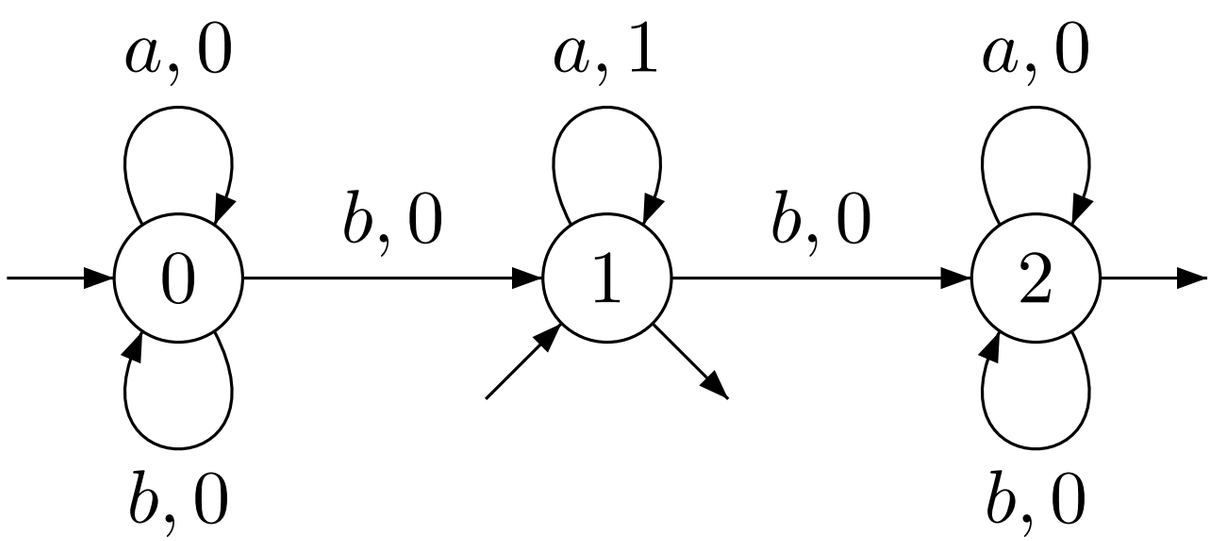

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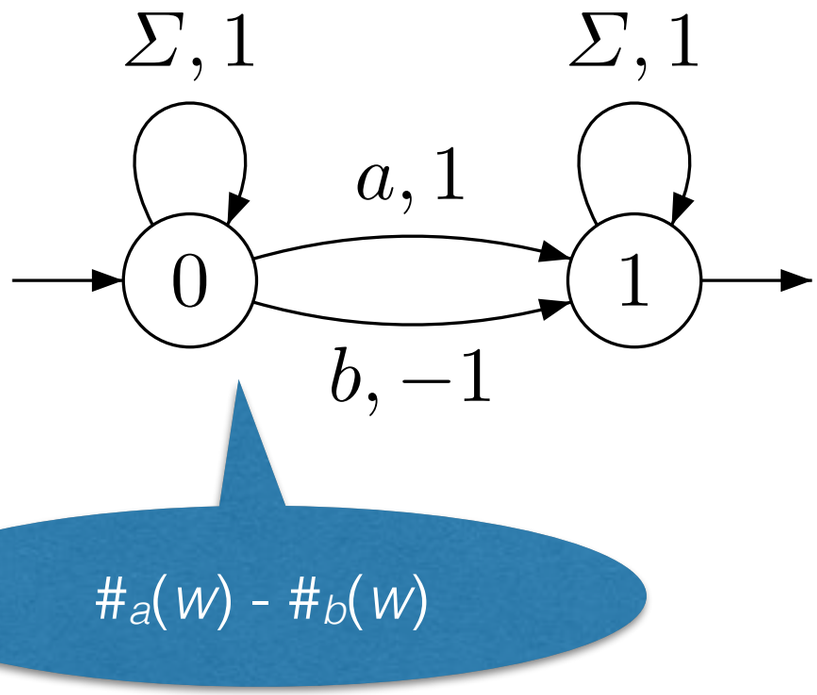
Semantics of *aaba*: max(2,1) = 2

Weighted Automata



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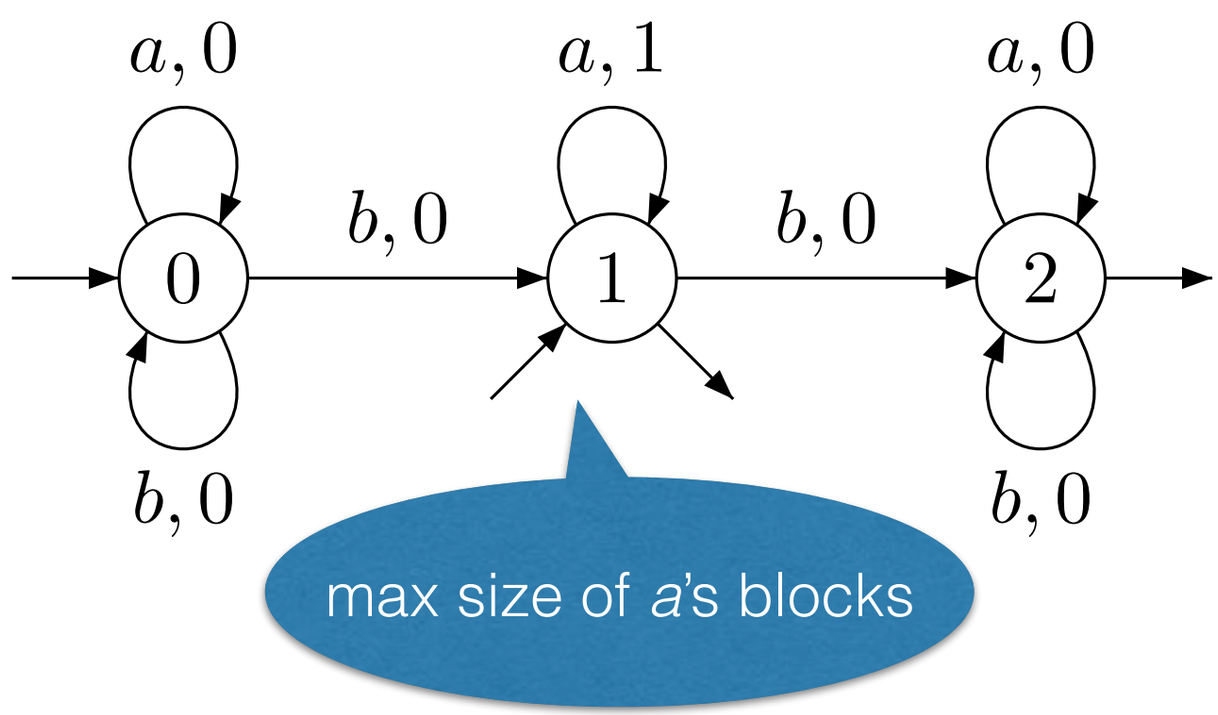

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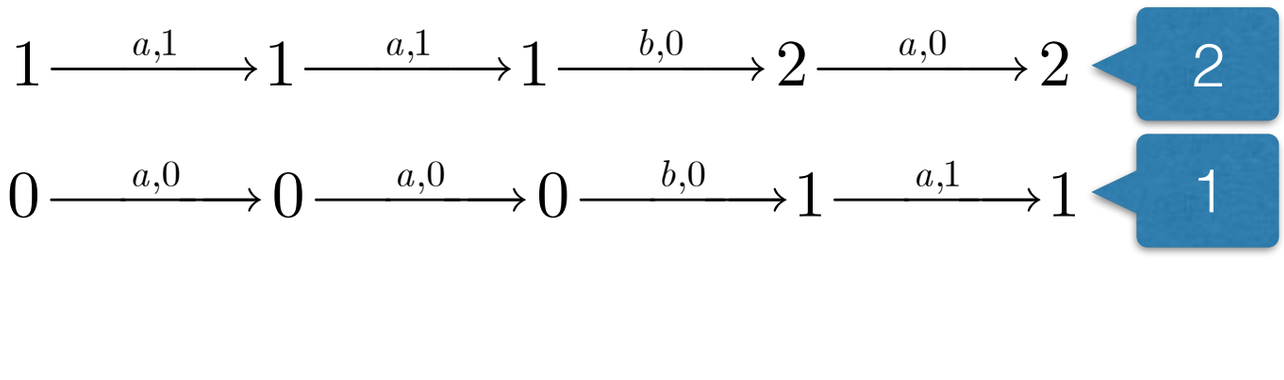

Semantics of *aaba*: max(2,1) = 2

Transductions

Transductions as weights

- Desire: weight transitions with words… Difficult to equip A* with a semiring structure: how to combine several accepting runs?

- Works for deterministic or unambiguous automata: functional transducers

- For relations: semiring of languages

$$(2^{A^*}, \cup, \cdot, \varnothing, \{\varepsilon\})$$

How to specify and decide properties of transductions (relations with infinite image)?

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Question: $|I| \subseteq |S|$ and $dom(I)=dom(S)$?

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Undecidable

Input: functional/deterministic transduction I, unambiguous/1-valued transduction S
Question: $|I| \subseteq |S|$ and $dom(I)=dom(S)$?

Input: (unambiguous/1-valued) transduction T
Question: synthesise, if possible, an equivalent functional/deterministic transduction T'

How to Specify Quantitative Properties?

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Weighted Monadic Second Order Logic [Droste&Gastin 05] generalized to trees [Droste&Vogler 06], infinite words [Droste&Rahonis 07], nested words [Mathissen 10] or pictures [Fichtner 11]

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Weighted Temporal Logics:
PCTL [Hansson&Jonsson 94], WLTL [Mandrali 12]

- Core weighted logic for weighted automata
- Enhancing the logic to handle more properties: FO vs pebbles
- A special case: the transducers

Monadic Second Order Logic (MSO)

$$\varphi ::= \top \mid P_a(x) \mid x \leq y \mid x \in X \mid \neg\varphi \mid \varphi \wedge \varphi \mid \forall x\ \varphi \mid \forall X\ \varphi$$

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$$\varphi_3 = \exists X \forall x \forall y\ \ (\varphi_{first}(x) \Rightarrow x \in X) \wedge$$
$$(y = x+1 \Rightarrow (x \in X \Leftrightarrow y \notin X)) \wedge (\varphi_{last}(x) \Rightarrow x \notin X)$$

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$\varphi_1 = \exists x\, P_a(x)$ « There is a letter a in the word »

$\varphi_2 = \forall x \exists y\, (y \leqslant x \wedge P_a(y))$ « The first letter of the word is a. »

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« The word has even length. »

Weighted MSO

$$\varphi ::= s \mid P_a(x) \mid x \leqslant y \mid x \in X \mid \neg P_a(x) \mid \neg(x \leqslant y) \mid \neg(x \in X)$$
$$\mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \exists x\, \varphi \mid \forall x\, \varphi \mid \exists X\, \varphi \mid \forall X\, \varphi$$

[Droste-Gastin, 05]

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Negation restricted to atomic formulae

[Droste-Gastin, 05]

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Arbitrary constants from a semiring

Negation restricted to atomic formulae

[Droste-Gastin, 05]

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- Semantics in a semiring $\mathbb{S} = (S, +, \times, 0, 1)$
  - Atomic formulae: 0, 1
  - disjunction, existential quantifications: sum
  - conjunction, universal quantifications: product
- Inspired from the boolean semiring $\mathbb{B} = (\{0, 1\}, \vee, \wedge, 0, 1)$

[Droste-Gastin, 05]

Weighted MSO

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$$[\![\varphi_1]\!](w) = |w|_a$$

[Droste-Gastin, 05]

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- Examples

$$\varphi_1 = \exists x\, P_a(x) \qquad \varphi_2 = \forall x \exists y\, (y \leqslant x \wedge P_a(y))$$

$$[\![\varphi_1]\!](w) = |w|_a \qquad [\![\varphi_2]\!](abaaab) = 1 \times 1 \times 2 \times 3 \times 3$$

$$[\![\varphi_2]\!](a^n) = n!$$

[Droste-Gastin, 05]

Weighted MSO

$$\varphi ::= s \mid P_a(x) \mid x \leqslant y \mid x \in X \mid \neg P_a(x) \mid \neg(x \leqslant y) \mid \neg(x \in X)$$
$$\mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \exists x\, \varphi \mid \forall x\, \varphi \mid \exists X\, \varphi \mid \forall X\, \varphi$$

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Too big to be computed by a weighted automaton

[Droste-Gastin, 05]

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We need to restrict weighted MSO

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commutativity

φ almost boolean

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- Boolean fragment

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$$P_a(x) \,?\, 1 : (P_b(x) \,?\, -1 : 0)$$

$$x \in X_1 \,?\, s_1 : (x \in X_2 \,?\, s_2 : \cdots (x \in X_{n-1} \,?\, s_{n-1} : s_n) \cdots)$$

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$$[\![\Psi]\!](w, \sigma) = s$$

if ... then ... else ...

some value occurring in Ψ

[Gastin-Monmege, 15]

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P_a

$x \in$ $X_2 : s_2 : \cdots (x \in X_{n-1}\,?\,s_{n-1} : s_n) \cdots)$

$[\![\Psi]\!](w, \sigma) = s$

A step formula takes finitely many values
For each value, the pre-image is MSO-definable

some value occurring in Ψ

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Assigns a value from Ψ to each position

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$$\{\!|\textstyle\prod_x \Psi|\!\}(w, \sigma) = \{\{(\llbracket\Psi\rrbracket(w, \sigma[x \mapsto i]))_i\}\} \in \mathbb{N}\langle R^\star\rangle$$

singleton multiset

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- Semantics
  - $\{\!|\, \mathbf{0} \,|\!\}(w, \sigma) = \varnothing$
  - sums over multisets $\quad \{\!|\, \Phi_1 + \Phi_2 \,|\!\}(w, \sigma) = \{\!|\, \Phi_1 \,|\!\}(w, \sigma) \uplus \{\!|\, \Phi_2 \,|\!\}(w, \sigma)$

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$$\Phi = \sum_X \sum_y \varphi_{block}(X,y) ? \prod_x (x \in X ? 1 : 0) : \mathbf{0}$$

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$$\{\!| \Phi |\!\}(baabab) = \{\{000000, 011000, 000010, 000000\}\}$$

Multisets of weight structures

- A run generates a sequence of weights $\mathsf{wgt}(\rho) = s_1 s_2 \cdots s_n$
- Abstract semantics $\{\!|\mathcal{A}|\!\}(w) = \{\!\{\mathsf{wgt}(\rho) \mid \rho \text{ run on } w\}\!\}$

multiset

[Gastin-Monmege, 15][Droste-Perevoshchikov, 14]

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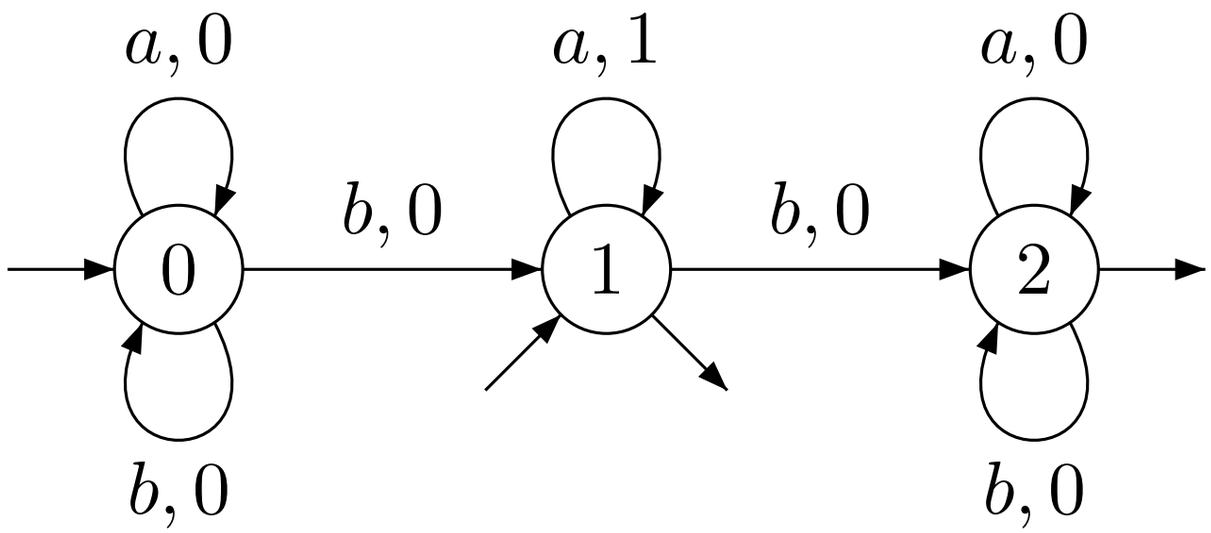

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- Aggregation $\mathsf{aggr} : \mathbb{N}\langle R^\star \rangle \to S$

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Multisets of weight structures

Semiring: sum-product

$$\mathsf{aggr}_{\mathsf{sp}}(A) = \sum \prod A = \sum_{r_1 \cdots r_n \in A} r_1 \times \cdots \times r_n$$

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- Aggregation

Average value
Discounted value…

$\rightarrow S$

[Gastin-Monmege, 15][Droste-Perevoshchikov, 14]

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Valuation structure: evaluator-valuation

$$\mathsf{aggr}_{\mathsf{ef}}(A) = F(\mathrm{Val}(A)) \text{ with } F \colon \mathbb{N}\langle U \rangle \to S \text{ and } \mathrm{Val} \colon U^\star \to U$$

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Theorem: weighted automata = core wMSO

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Theorem: weighted automata = core wMSO

- Abstract semanti[illegible]
- Concrete [illegible] $\to S$

Easy constructive proofs
preservation of the constants
no restriction on core wMSO
no hypotheses on weights

[Gastin-Monmege, 15]

Extensions

More general models than words:
trees, nested words…

More powerful logics:
deciding if a wMSO formula is expressible in core wMSO?

More powerful automata: finding equivalent fragments of wMSO

Weighted FO logic

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Reintroduction of the product

Unconditional product quantification

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Reintroduction of the product

Unconditional product quantification

$$\varphi_2 = \prod_x \sum_y (x \leq y \wedge P_a(x)) ? 1 : 0 \qquad [\![\varphi_2]\!](a^n) = n!$$

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Reintroduction of the product

Unconditional product quantification

$$[\![\prod_x \prod_y 2]\!](w) = 2^{|w|^2}$$

Pebble weighted automata

$$I \in \mathbb{S}^Q \qquad T \in \mathbb{S}^Q$$

$$\mathcal{A} = (Q, A, I, \delta, T)$$

$$\delta \colon Q \times \mathrm{Test} \times \mathrm{Move} \times Q \to \mathbb{S}$$

$$\mathrm{Move} = \{\rightarrow, \leftarrow, \uparrow\} \cup \{\downarrow_x \mid x \in \mathrm{Peb}\}$$

Run as a finite sequence of configurations (W, σ, q, i, π)

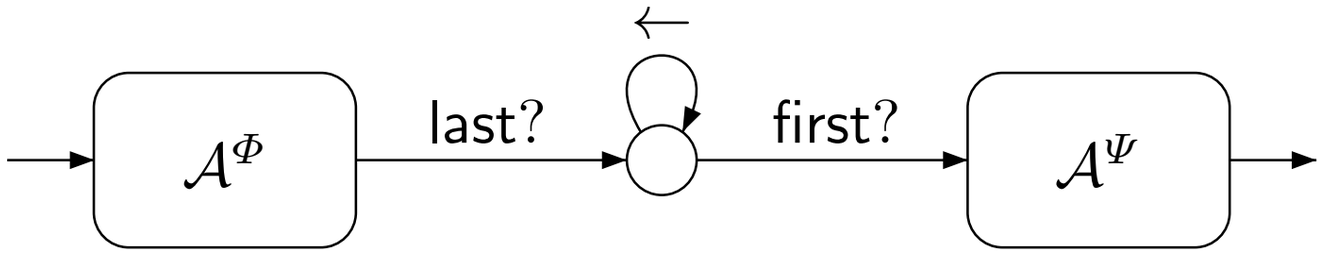
with free pebbles $\sigma \colon \mathrm{Peb} \to \mathrm{pos}(W)$

and a stack of currently dropped pebbles $\pi \in (\mathrm{Peb} \times \mathrm{pos}(W))^*$

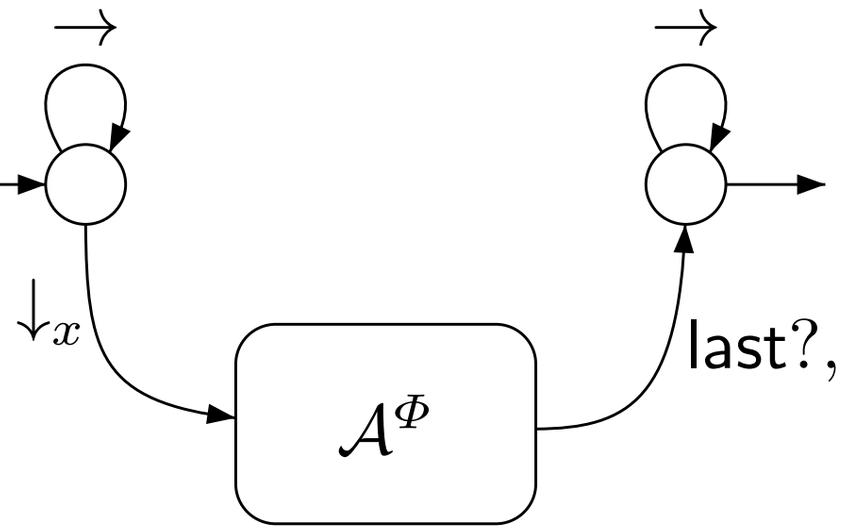
Translating a formula into an automaton

Sum by disjoint union of automata

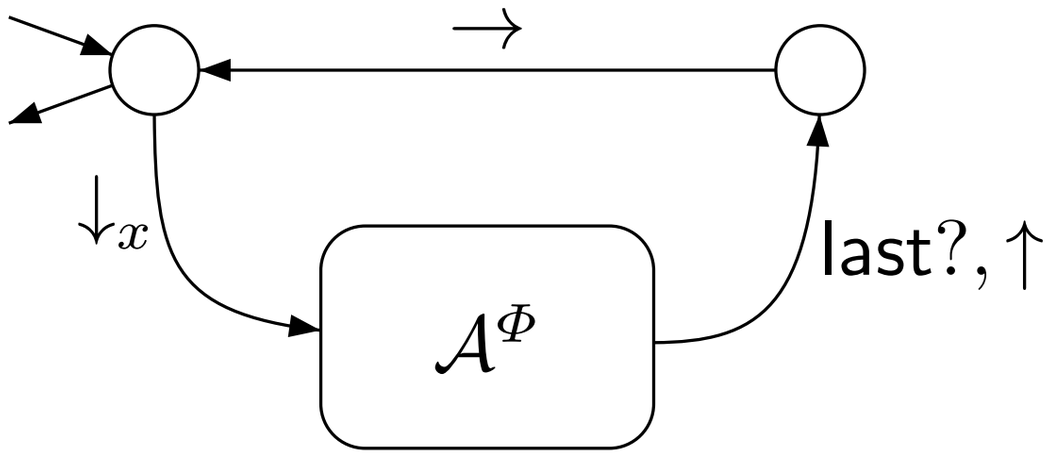
Product:



Sum quantification:



Product quantification:



Translating a formula into an automaton

Challenging for the *Boolean part*:
need unambiguous automata

Translating a formula into an automaton

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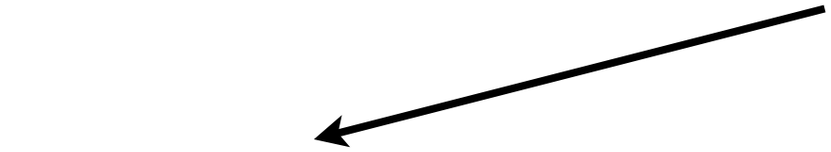

Use deterministic automata
of size non-elementary...

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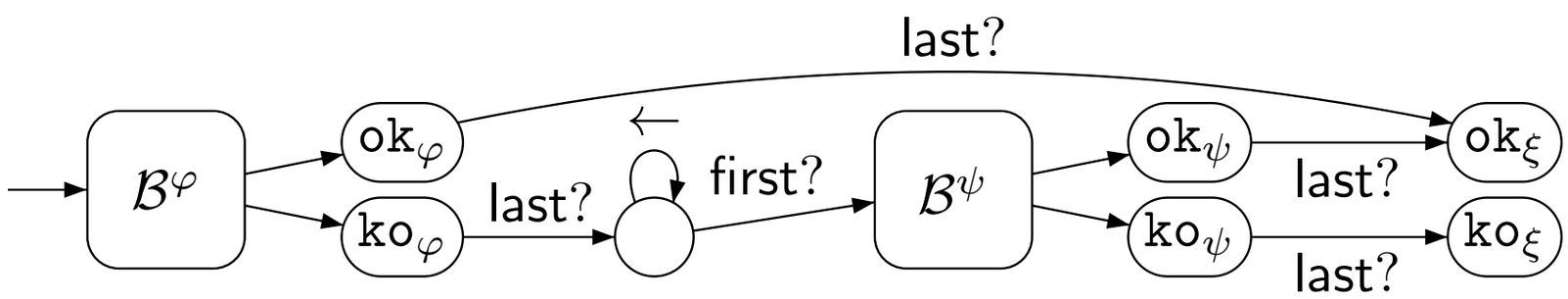
$\xi = \varphi \vee \psi$



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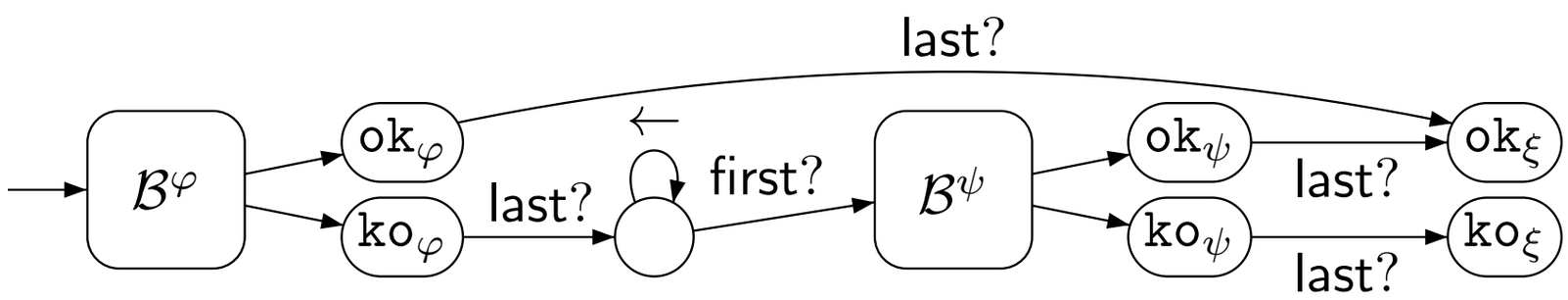
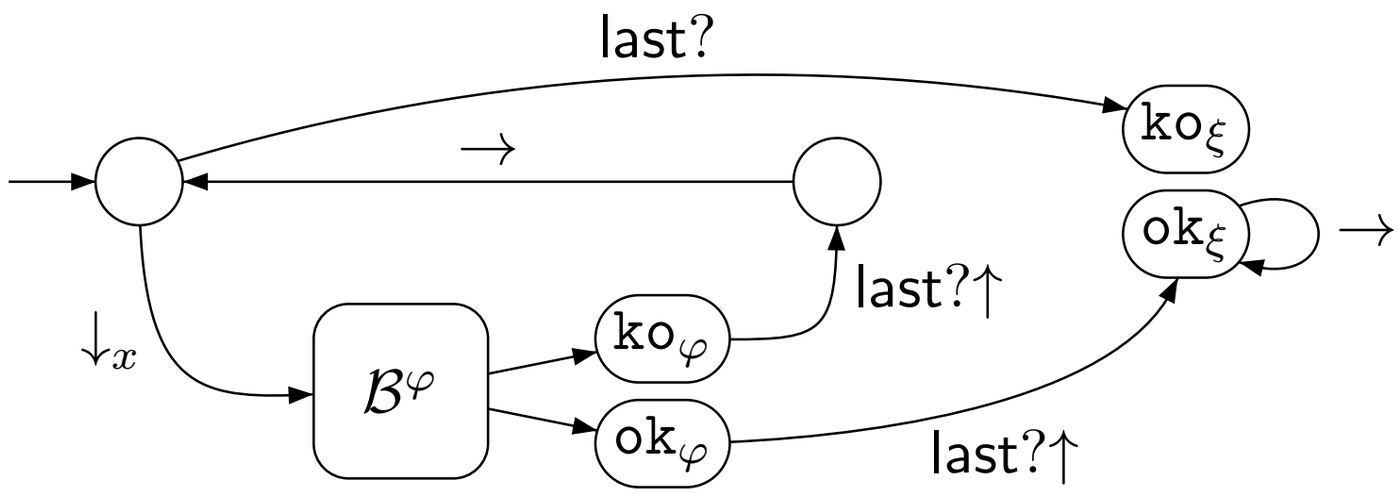


Existential/Universal quantifications

$\xi = \exists x\, \varphi$



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- Solution: weighted transitive closure operation

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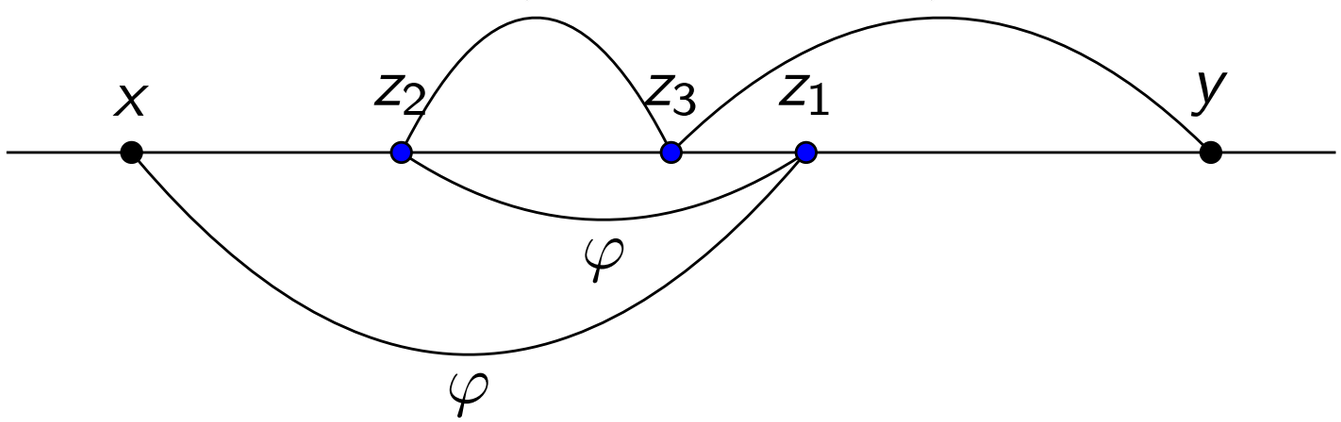

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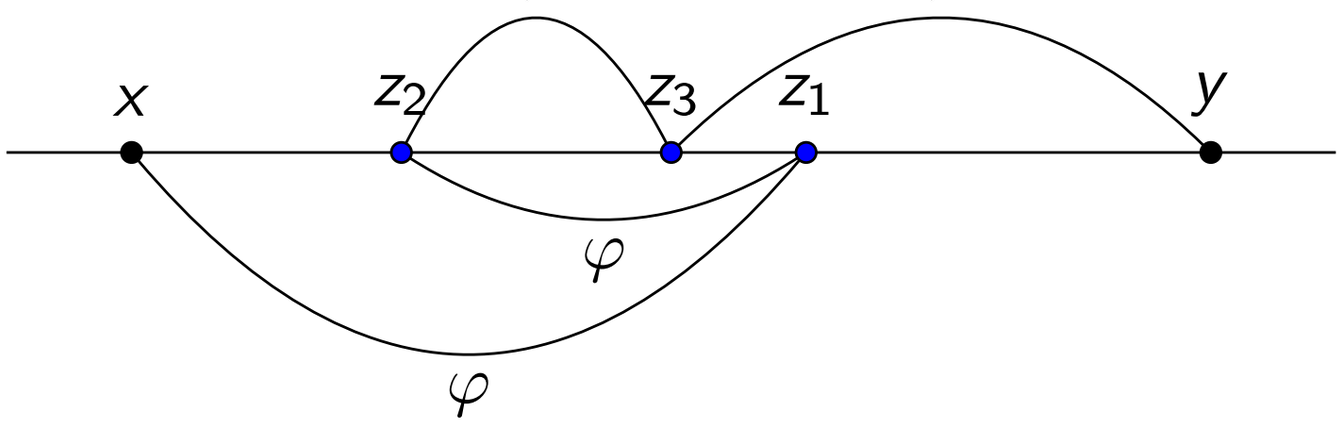

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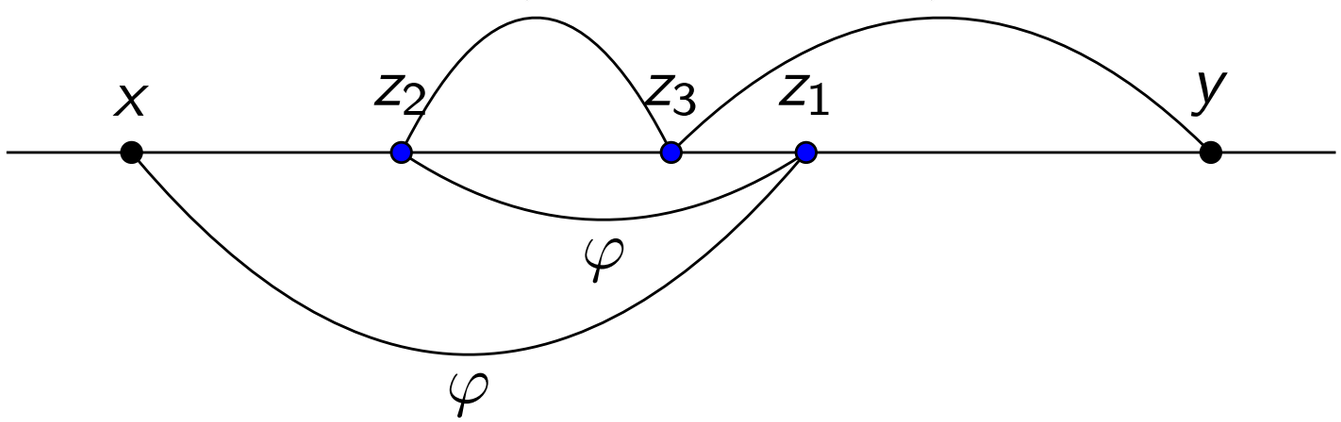
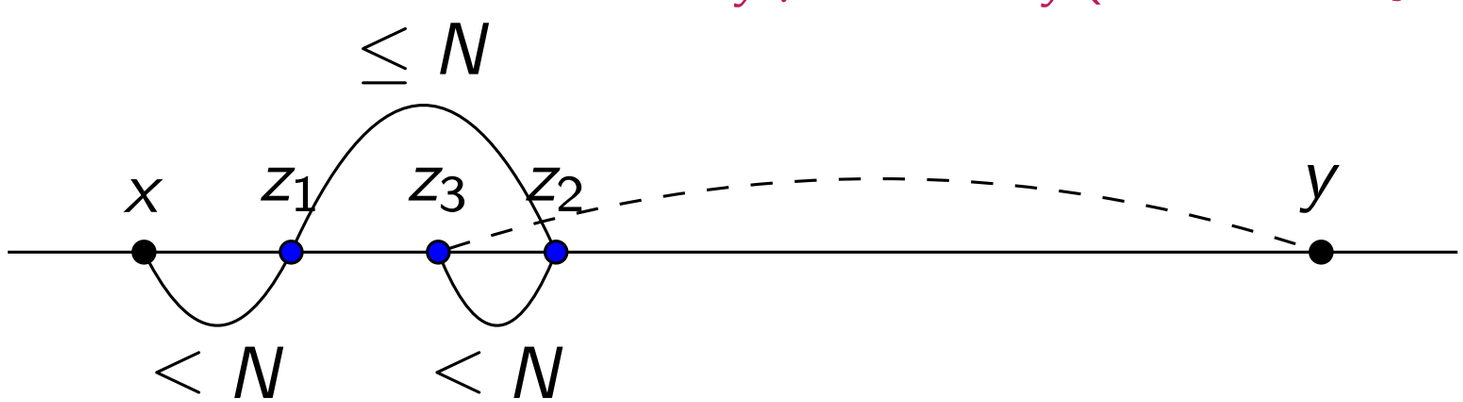


Bounded transitive closure : $N\text{-}\mathsf{TC}_{xy}\varphi = \mathsf{TC}_{xy}(x - N \le y \le x + N \wedge \varphi)$



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Theorem: PWA = wFO + bounded-TC

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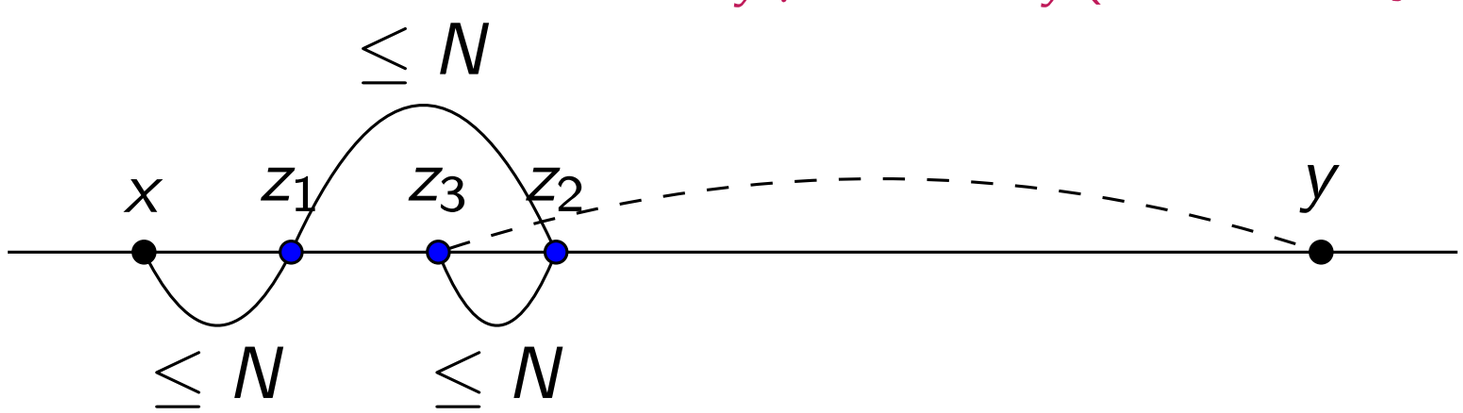

(Partial) conclusion

core wMSO = weighted automata

wFO + wTC = pebble navigating weighted automata

(Partial) conclusion

core wMSO = weighted automata

wFO + wTC = pebble navigating weighted automata

True for semirings, but other weight domains also!

average, discounted sums, ratio, energy…

True for finite words, but other input structures also!

(un)ranked trees, nested words, grids,
arbitrary graphs, infinite structures…

How to specify and decide properties of transductions with infinite image?

Input: functional/deterministic transduction I (implementation = transducer), transduction S (specification = transducer or formula)
Question: $|I| \subseteq |S|$ and $dom(I)=dom(S)$?

Undecidable

Input: functional/deterministic transduction I, unambiguous/1-valued transduction S
Question: $|I| \subseteq |S|$ and $dom(I)=dom(S)$?

Input: (unambiguous/1-valued) transduction T
Question: synthesise, if possible, an equivalent functional/deterministic transduction T'

$$\left(2^{A^*}, \cup, \cdot, \varnothing, \{\varepsilon\}\right)$$

Example of transductions

$$\prod_x (P_x(a)?\{aa\} : (P_x(b)?\{bb\} : \varnothing))$$

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$$a\star b\star a \longmapsto \{ainsertba, abinserta\}$$

Relation

Examples

$$\prod_x P_x(a)?\{a\} : (P_x(b)?\{\varepsilon\}) \times \prod_x P_x(a)?\{\varepsilon\} : (P_x(b) : \{c\})$$

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Not comp. by 1-way Func Transducer

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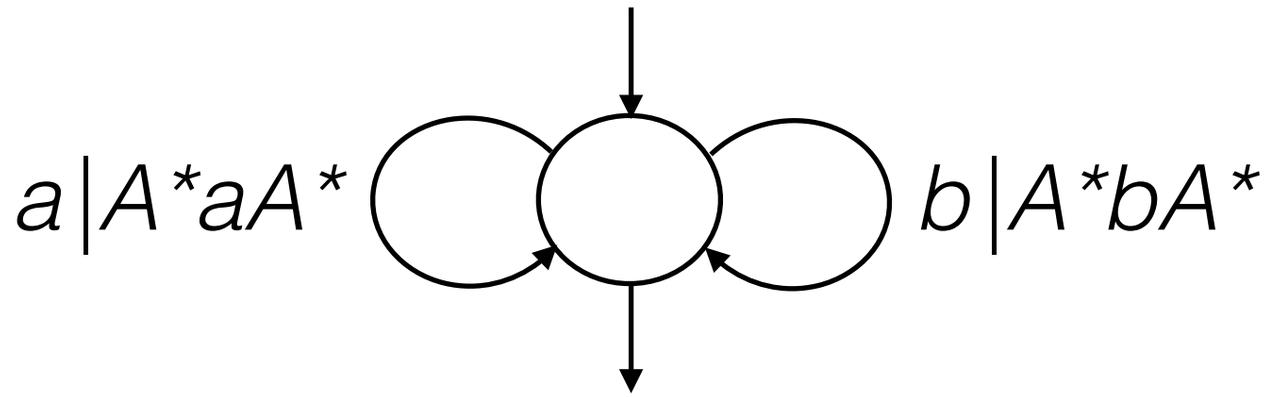
Infinitely-valued relation

$$\prod_x P_x(a)?\,A^*aA^* : (P_x(b)?\,A^*bA^*)$$

aba $\longmapsto$ *A*aA*bA*aA**

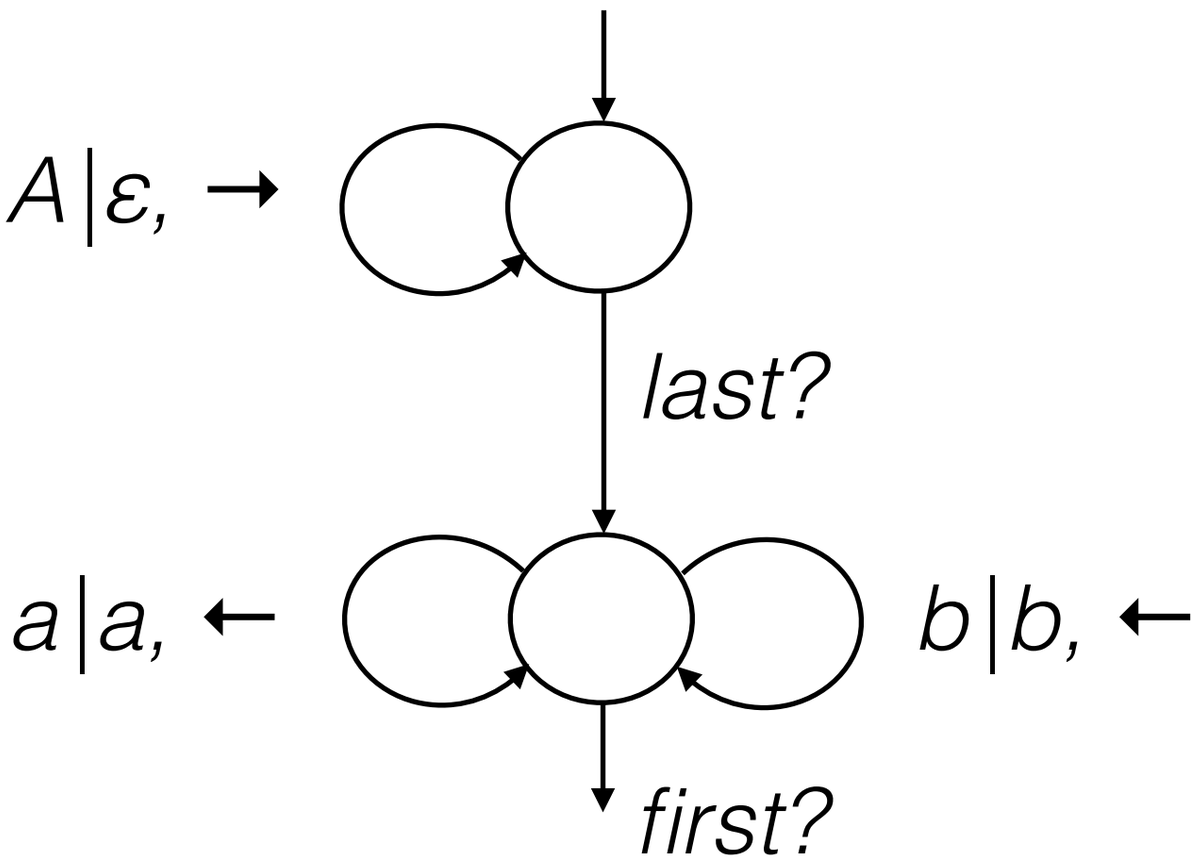
Transducers

$$\prod_x P_x(a)?\, A^* a A^* : (P_x(b)?\, A^* b A^*)$$



Infinite-valued, but <u>deterministic</u>

Reverse?



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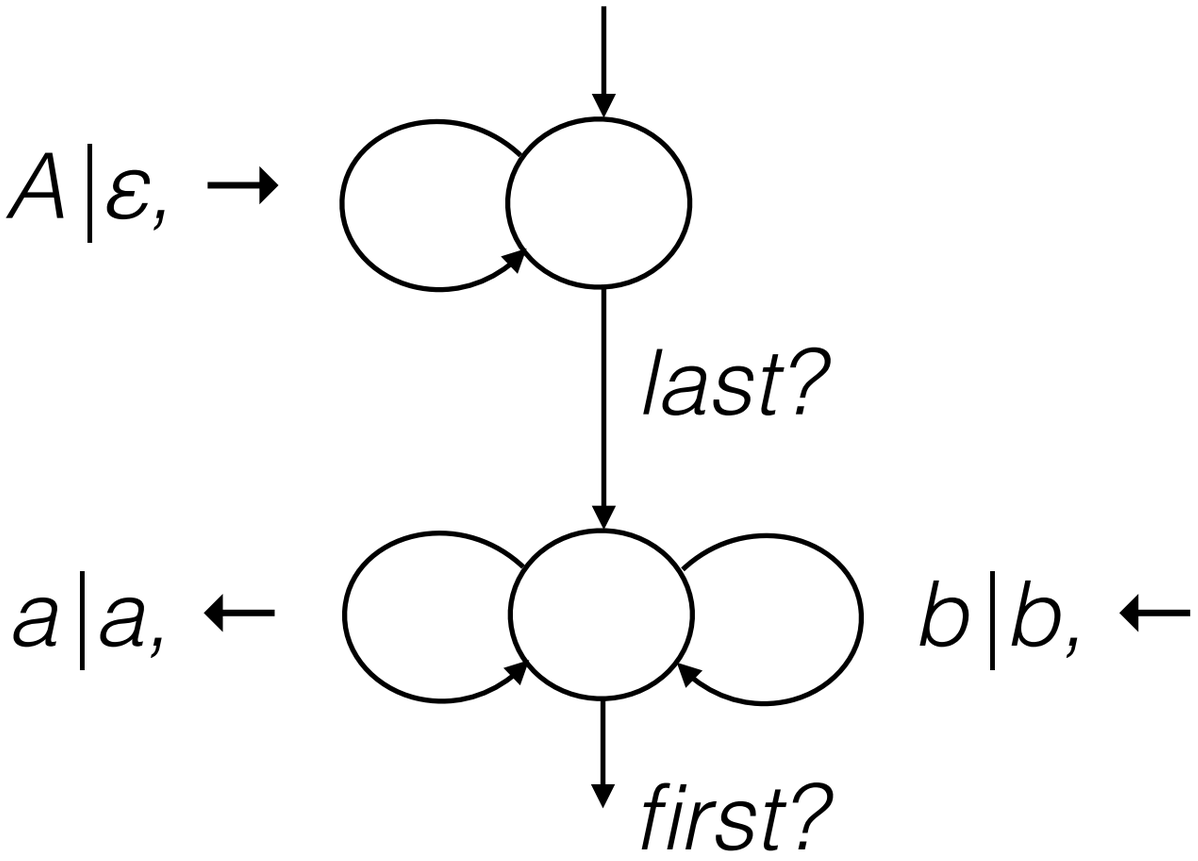

Impossible in FO…
… because of order of interpretation of product

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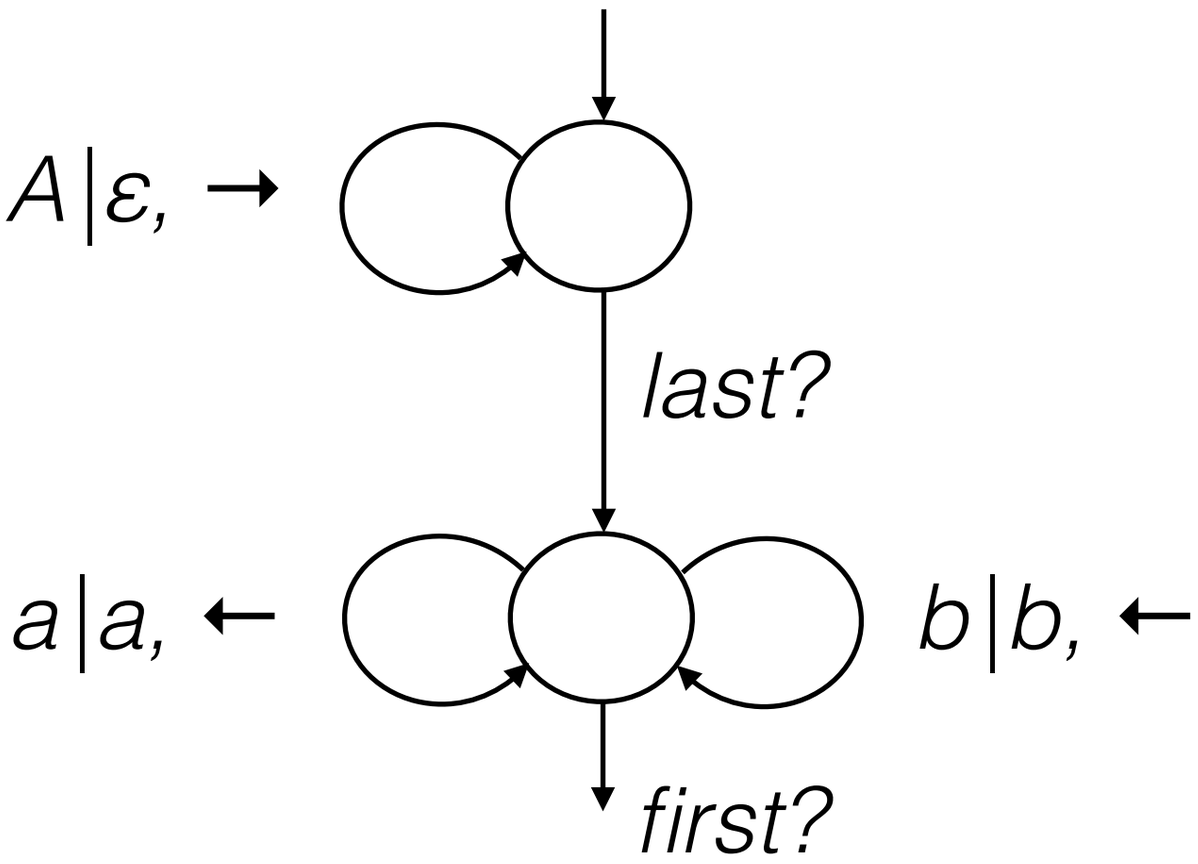

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Solution: in this non-commutative setting,
add right-to-left products

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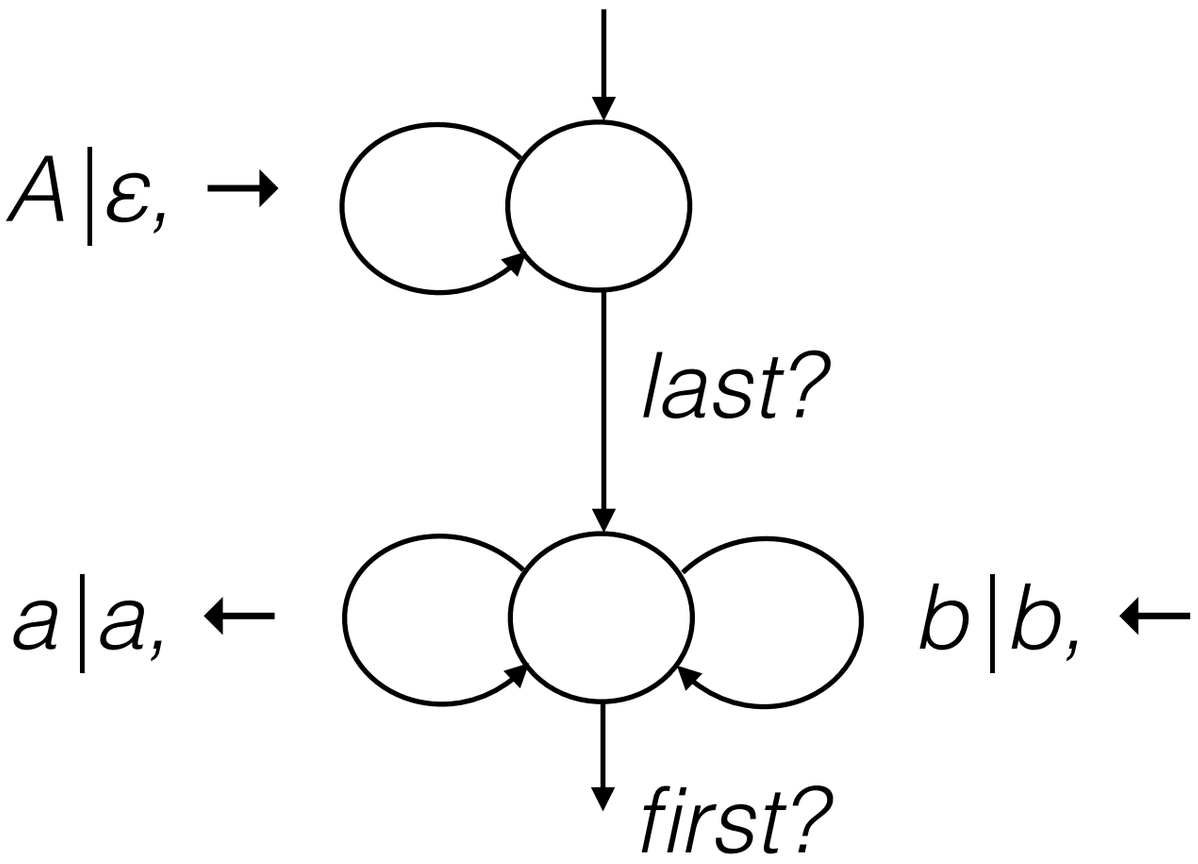

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$$\varphi ::= \top \mid P_a(x) \mid x \leqslant y \mid x \in X \mid \neg\varphi \mid \varphi \wedge \varphi \mid \forall x\, \varphi$$

$$\Phi ::= L \mid \varphi ? \Phi : \Phi \mid \Phi + \Phi \mid \Phi \times \Phi \mid \sum_x \Phi \mid \prod_x \Phi \mid N\text{-}TC_{x,y}\Phi$$

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Regular language

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Regular language

Able to define right-to-left product

$$\coprod\nolimits_x \Phi(x) \coloneqq [1\text{-}TC_{x,y}(y = x - 1 ? \Phi(x))](last, first) \times \Phi(first)$$

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Theorem: Pebble Transducers = FO + bounded-TC

with regular language productions

linear transformation from logic to transducers

Algorithmic questions

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Theorem: Evaluation of FO + bounded-TC with complexity O(|formula|x|input|$^{\#variables}$)

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Theorem: Evaluation of FO + bounded-TC with complexity O(|formula|x|input|$^{\#variables}$)

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  - Extension to the general setting?…

Thank you!