

The Fatou property of submonoids of $A^* \times B^*$

Luc Boasson and Olivier Carton

IRIF – CNRS and Université Paris Diderot

ANR DeLTA – Bordeaux

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Outline

The problem

(Well) Known results

What's new

Nasty examples

Technics

Motivation



Global vs Local constraints

Global constraints of input/output: **monoid** $M \subseteq A^* \times B^*$

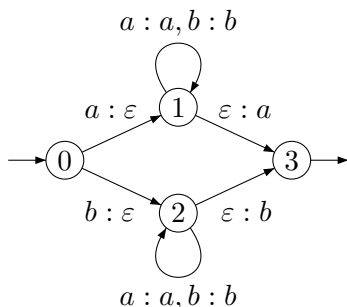
A relation $R \subseteq M$ realized by some **transducer**.

Length-preserving function

$$\{a, b\}^* \rightarrow \{a, b\}^*$$

$$a_1 a_2 a_3 \cdots a_n \mapsto a_2 a_3 \cdots a_n a_1$$

Here $M = \{(u, v) : |u| = |v|\}$



Question: can R be realized by a transducer with labels in M ?

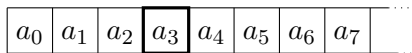
This question can be rephrased as the following **Fatou property**:

$$R \in \text{Rat}(A^* \times B^*) \wedge R \subseteq M \stackrel{?}{\implies} R \in \text{Rat}(M)$$

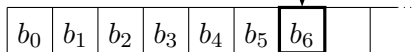


Transducers

Input tape

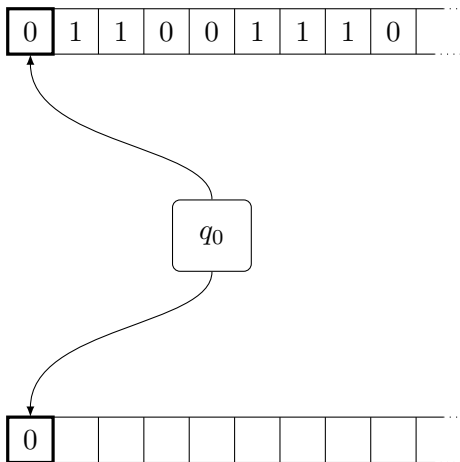
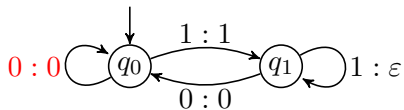


Output tape

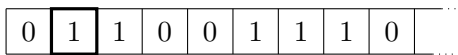
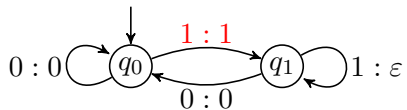


Transitions $p \xrightarrow{a:v} q$ for $a \in A, v \in B^*$.

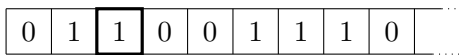
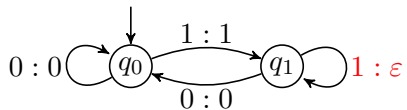
Example



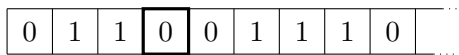
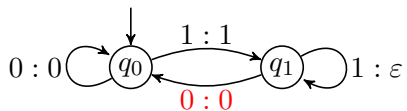
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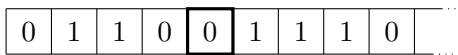
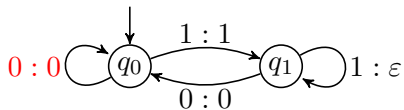
Example



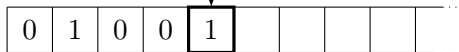
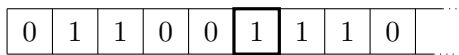
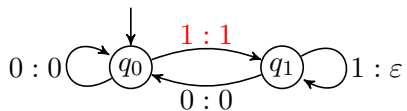
Example



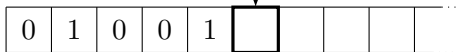
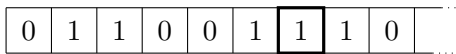
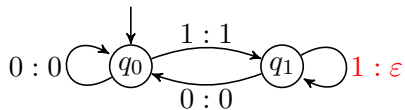
Example



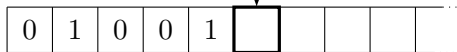
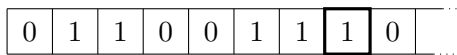
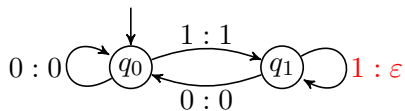
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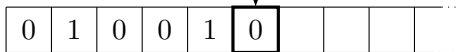
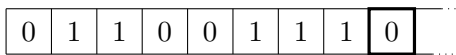
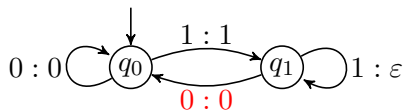
Example



Example



Example



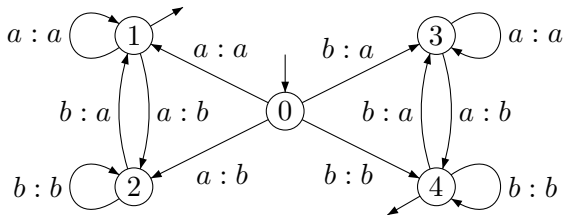
Length preserving relations

$$M = \{(u, v) : |u| = |v|\}$$

Theorem (Elgot & Mezei 65)

The following conditions are equivalent for a rational relation R

- ▶ $R \subseteq M$
- ▶ R is realized by a transducer such that each transition $p \xrightarrow{u:v} q$ satisfies $(u, v) \in M$, that is $|u| = |v|$.



Length decreasing relations

$$M = \{(u, v) : |u| \geq |v|\}$$

Theorem (Leguy 81)

The following conditions are equivalent for a rational relation R

- ▶ $R \subseteq M$
- ▶ R is realized by a transducer such that each transition $p \xrightarrow{u:v} q$ satisfies $(u, v) \in M$, that is $|u| \geq |v|$.

Key ingredients:

- ▶ The condition is satisfied for cycles.
- ▶ The delay is therefore bounded.
- ▶ The synchronized transducer uses two buffers of bounded size.



Stable monoids

A submonoid of \mathbb{N}^k is called **stable** if one of the two equivalent statements hold.

- ▶ $N = C \cap H$ where C closed cone of \mathbb{N}^k and H subgroup of \mathbb{Z}^k
- ▶ $\forall u \in N, \forall v \in \mathbb{N}^k \quad u + v^\oplus \subseteq N \implies v \in N$

Theorem (Sakarovitch)

If $M = \mu^{-1}(N)$ where $\mu : A^* \times B^* \rightarrow \mathbb{N}^k$ and N stable monoid of \mathbb{N}^k , then M has the Fatou property.

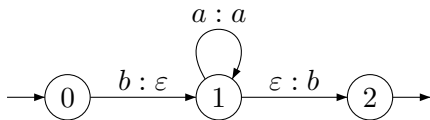
$$R \in \text{Rat}(A^* \times B^*) \wedge R \subseteq M \implies R \in \text{Rat}(M)$$



Negative example

Let M be the monoid

$$M = \{(a, \varepsilon), (\varepsilon, a), (b, b)\}^* = \{(u, v) : |u|_b = |v|_b\}$$



This transducer realizes the relation R given by

$$R = \{(ba^n, a^n b) : n \geq 0\} \subset M.$$

This relation cannot be realized by a transducer over M .
The monoid M does not have the Fatou property.



Quasi-unary monoids

A monoid $M \subseteq A^* \times B^*$ is **quasi-unary** if

$$(u, v) \in M \wedge |v| = |v'| \implies (u, v') \in M.$$

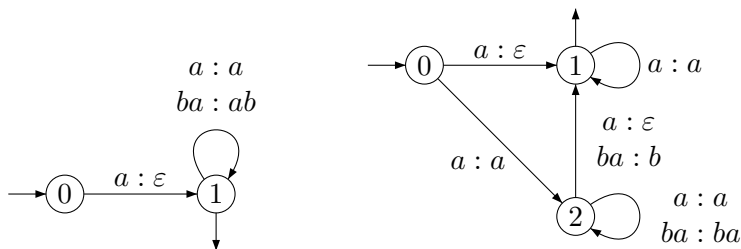
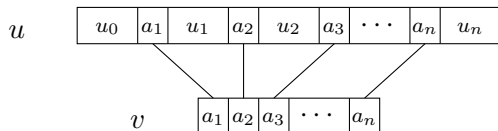
Theorem

A quasi-unary monoid generated by an alphabetic set that is contained in $A \times B^$, has the Fatou property.*



The subword monoid

A word $v = a_1 \cdots a_n$ is a **subword** of a word u , denoted $v|u$, if there exist $n + 1$ words u_0, \dots, u_n such that $u = u_0 a_1 u_1 \cdots a_n u_n$.



Theorem

The monoid $M = \{(u, v) : v|u\}$ has the Fatou property.



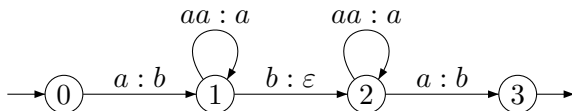
Nasty examples

Let $\mu : A^* \rightarrow B^*$ be a morphism. The graph of μ

$$M = \{(u, \mu(u)) : u \in A^*\}$$

has the Fatou property. What about finite substitutions ?

$$M = \{(a, a), (a, b), (a, \varepsilon), (b, b)\}^*$$



$$R = \{(a^{2m+1}ba^{2n+1}, ba^{m+n}b) : m, n \geq 0\}$$

The relation R satisfies $R \subset M$ since $m + n \leq \max(2m, 2n)$ but it cannot be realized by a transducer with labels in M .

$$M = \{(a, a), (a, \varepsilon), (b, b), (b, a)\}^*$$

Open question: does this monoid M have the Fatou property ?



Delays

A run ρ of label $(u, v) = (u_1 \cdots u_n, v_1 \cdots v_n)$

$$q_0 \xrightarrow{u_1:v_1} q_1 \xrightarrow{u_2:v_2} q_2 \cdots q_{n-1} \xrightarrow{u_n:v_n} q_n$$

A M -factorization ϕ of (u, v)

$$(u, v) = (a_1, w_1) \cdots (a_m, w_m) \quad \text{where} \quad (a_i, w_i) \in M$$

The **delay between ρ and ϕ in q_i** is

$$d(\rho, \phi, i) = ||v_1 \cdots v_i| - |w_1 \cdots w_j|| \quad \text{where} \quad u_1 \cdots u_i = a_1 \cdots a_j$$

The **delay between ρ and ϕ** is $d(\rho, \phi) = \max_{1 \leq i \leq n} d(\rho, \phi, i)$.

The **delay of ρ** is $d(\rho) = \min_{\phi} d(\rho, \phi)$

The **delay of the transducer** is $d(\mathcal{T}) = \max_{\rho} d(\rho)$



Delays (continued)

Proposition

Any transducer \mathcal{T} of bounded delay n such that $|\mathcal{T}| \subseteq M$ is equivalent to the transducer \mathcal{T}_n over M .

For all known example ($M_=$, M_{\leq} , subword monoid), it is shown that if transducer \mathcal{T} satisfies $|\mathcal{T}| \subseteq M$, then its delay must be bounded. What about the converse ?

Proposition

Let M be a fixed monoid. If there exists a computable function f such that for any transducer \mathcal{T} , $|\mathcal{T}| \subseteq M$ implies that the delay of \mathcal{T} is bounded by $f(\mathcal{T})$, then, it is decidable whether $|\mathcal{T}| \subseteq M$.



Non-finitely generated monoid

For any non-negative real number α , let M_α

$$M_\alpha = \{(u, v) : |u| \leq \alpha|v|\}$$

Proposition

For any irrational number α , the monoid M_α is not finitely generated but it has the Fatou property.

A rational non-finitely generated cannot have the Fatou property.



The weak Fatou property

A monoid M has the **weak Fatou property** if

$$R \in \text{Rat}(A^* \times B^*) \wedge R \subseteq M \implies R \in \text{Rat}(\text{Rec}(M))$$

Proposition

The monoid $M = \{(\varepsilon, \varepsilon)\} \cup \{(u, v) : |u| \neq 0\}$ has the weak Fatou property.

For any non-negative real number α , let M_α

$$M'_\alpha = \{(u, v) : |u| < \alpha|v|\}$$

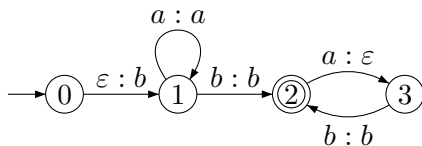
Proposition

For any rational number α , the monoid M'_α is not finitely generated and it does not have the weak Fatou property.



Infinite words

$$R = \{(a^n (ba)^\omega, ba^n b^\omega) : n \geq 0\} = (\varepsilon, b)(a, a)^*(ba, b)^\omega.$$



For each integer n , the output ω -word $ba^n b^\omega$ is a subword of the input ω -word $a^n (ba)^\omega$. However, this relation cannot be realized by a transducer with labels in $M = \{(u, v) : v|u\}$.

Normality (Borel 1909)

The number of **occurrences** of a word u in a word w is

$$\text{occ}(w, u) = \#\{i : w[i..i + |u| - 1] = u\}$$

An infinite word $x \in A^\omega$ (resp. a real number ξ) is **simply normal** (in base b) if for any $a \in A$,

$$\lim_{n \rightarrow \infty} \frac{\text{occ}(x[1..n], a)}{n} = \frac{1}{b}.$$

An infinite word $x \in A^\omega$ (resp. a real number ξ) is **normal** (in base b) if for any $u \in A^*$,

$$\lim_{n \rightarrow \infty} \frac{\text{occ}(x[1..n], u)}{n} = \frac{1}{b^{|u|}}.$$

In base $b = 2$, this means

- ▶ the frequencies in x of the 2 digits 0 and 1 are $1/2$,
- ▶ the frequencies in x of the 4 words 00, 01, 10, 11 are $1/4$,
- ▶ the frequencies in x of the 8 words 000, 001, ..., 111 are $1/8$,
- ▶ ...



Selection rules

- ▶ If $x = a_1a_2a_3 \cdots$ is a normal infinite word, then so is $x' = a_2a_3a_4 \cdots$ made of symbols at all positions but the first one.
- ▶ If $x = a_1a_2a_3 \cdots$ is normal infinite word, then so is $x' = a_2a_4a_6 \cdots$ made of symbols at even positions.
- ▶ What about selecting symbols at positions 2^n ?
- ▶ What about selecting symbols at prime positions ?
- ▶ What about selecting symbols following a 1 ?
- ▶ What about selecting symbols followed by a 1 ?

Prefix selection

Let $L \subseteq A^*$ be a set of finite words and $x = a_1 a_2 a_3 \cdots \in A^\omega$.

The **prefix selection** of x by L is the word $x \upharpoonright L = a_{i_1} a_{i_2} a_{i_3} \cdots$ where $\{i_1 < i_2 < i_3 < \cdots\} = \{i : a_1 a_2 \cdots a_{i-1} \in L\}$.

Example (Symbols following a 1)

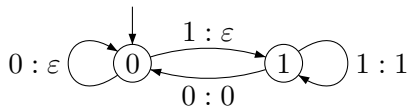
If $L = (0 + 1)^* 1$, then $i_1 - 1, i_2 - 1, i_3 - 1$ are the positions of 1 in x and $x \upharpoonright L$ is made of the symbols following a 1.

Theorem (Agafonov 1968)

Prefix selection by a rational set of finite words preserves normality.

The selection can be realized by a transducer.

Example (Selection of symbols following a 1)



Open questions

- ▶ Find a characterization of submonoids $M \subseteq A^* \times B^*$ having the Fatou property.
- ▶ For each submonoid $M \subseteq A^* \times B^*$ having the Fatou This naturally raises the second problem whether the Fatou property implies the decidability of the inclusion $R \subseteq M$.

