

How to prove that a language is regular or star-free?

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Part V

Logic

$$\varphi = \exists x \text{ ax}$$

$$L = A^*aA^*$$

$$\varphi = \exists x \exists y \exists z \ x < y \wedge y < z \wedge \text{ax} \wedge \text{by} \wedge \text{cz}$$

$$L = A^*aA^*bA^*cA^*$$

$$\varphi = \exists x (\forall y \neg y < x \wedge \text{ax})$$

$$L = aA^*$$

For each letter $a \in A$, let \mathbf{a} be a unary predicate symbol, where $\mathbf{a}x$ is interpreted as “the letter in position x is an a ”.

To each word $u \in A^+$ is associated a structure

$$\mathcal{M}_u = (\{1, 2, \dots, |u|\}, \mathbf{S}, <, (\mathbf{a})_{a \in A})$$

where $<$ is interpreted as the **usual order** on $\{1, 2, \dots, |u|\}$ and \mathbf{S} is the **successor relation**.

The language defined by a sentence φ is

$$L(\varphi) = \{u \in A^* \mid \mathcal{M}_u \text{ satisfies } \varphi\}$$

Logical fragments

Let $\mathbf{FO}[<]$, $\mathbf{MSO}[<]$ denote the set of **first-order** and **monadic second-order** formulas of signature $\{<, (\mathbf{a})_{a \in A}\}$, respectively.

Let $\mathbf{MSO}[S]$ and $\mathbf{SO}[S]$ denote the set of **monadic second-order** and **second-order** formulas of signature $\{S, (\mathbf{a})_{a \in A}\}$, respectively.

Theorem (Buchi 1960, Elgot 1961, Trakhtenbrot 1973)

MSO[S] captures the class of *regular languages*.

Theorem (McNaughton 1971)

FO[$<$] captures the class of *star-free languages*.

In general, *second order logic* captures some non-regular languages, but two successive results led to a complete characterisation of the syntactic fragments that only capture *regular languages*.

Existential second order

Existential second order (ESO or Σ_1^1)

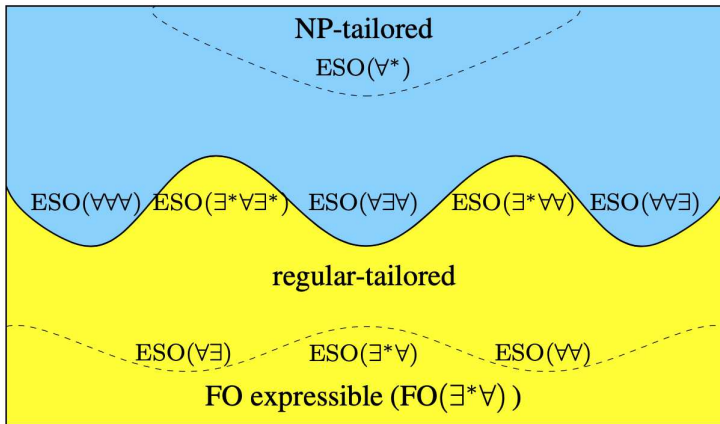
For example, $\text{ESO}(\exists^* \forall \exists^*)$ is the class of all formulas

$$\exists R_1 \cdots \exists R_n \exists x_1 \cdots \exists x_p \forall y \exists z_1 \cdots \exists z_q \\ \varphi(R_1, \dots, R_n, x_1, \dots, x_p, y, z_1, \dots, z_q)$$

where φ is quantifier-free and R_1, \dots, R_n are relation symbols.

ESO-prefix classes (pictures from EGS 2010)

Eiter, Gottlob, Gurevich 00 and Eiter, Gottlob, Schwentick 02



regular
 NP-hard

Beyond existential second-order

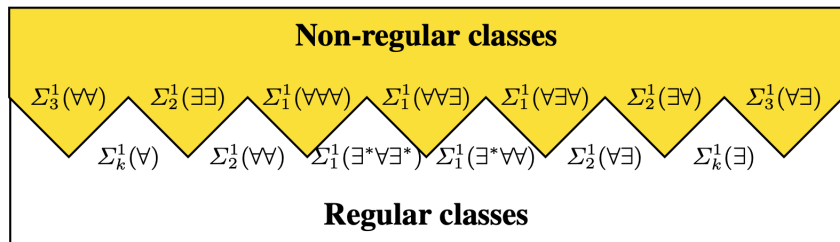


Fig. 2. Maximal regular and minimal non-regular SO prefix classes on strings

For instance $\Sigma_2^1(\forall\exists)$ corresponds to formulas

$$\exists R_1 \cdots \exists R_n \forall S_1 \cdots \forall S_m \forall x \exists y$$

$$\varphi(R_1, \dots, R_n, S_1, \dots, S_m, x, y)$$

where $R_1, \dots, R_n, S_1, \dots, S_m$ are **relations** and φ is **quantifier-free**.

Linear temporal logic (LTL)

The vocabulary consists of an atomic proposition p_a for each letter a , the usual connectives \vee , \wedge and \neg and the temporal operators **X** (next), **F** (eventually) and **U** (until).

Theorem (Kamp 1968)

*A language is **star-free** iff it is **LTL-definable**.*

Rabin's tree theorem

For each letter a , let S_a be a **binary relation symbol**, interpreted on A^* as follows: $S_a(u, v)$ iff $v = ua$.

Theorem (Rabin 1969)

*A language is **regular** iff it is definable in **MSO** $[(S_a)_{a \in A}]$.*

Part VI

Transductions and profinite topology

Let M and N be monoids. A **transduction** $\tau: M \rightarrow N$ is a **relation** on M and N , viewed as a function from M to $\mathcal{P}(N)$.

The **inverse transduction** $\tau^{-1}: N \rightarrow M$ is defined by $\tau^{-1}(Q) = \{m \in M \mid \tau(m) \cap Q \neq \emptyset\}$.

Recognizable subsets of a monoid

Let L be a subset of a monoid M . A monoid N recognizes L if there exists a surjective morphism $h : M \rightarrow N$ such that $L = h^{-1}(h(L))$.

Let $\text{Rec}(M)$ denote the set of recognizable subsets of M (= recognized by some finite monoid.).

For A^* , recognizable = regular = rational.

Transductions preserving recognizability

A function [transduction] $f : M \rightarrow N$ preserves recognizability if, for each recognizable subset R of N , $f^{-1}(R)$ is recognizable.

Proposition (Pin-Silva 2005)

The function $g : A^* \times \mathbb{N} \rightarrow A^*$ defined by $g(x, n) = x^n$ preserves recognizability.

Let $\tau_n : A^* \rightarrow (A^*)^n$ be defined by

$$\tau_n(u) = \{(u_1, \dots, u_n) \mid u_1 \cdots u_n = u\}$$

Then both τ_n and τ_n^{-1} preserve recognizability.

Residually finite monoids

A monoid N separates two elements x, y of a monoid M if there exists a monoid morphism $h : M \rightarrow N$ such that $h(x) \neq h(y)$.

A monoid M is residually finite if any pair of distinct elements of M can be separated by a finite monoid.

Let \mathcal{M} be the class of monoids that are finitely generated and residually finite. This class includes finite monoids, free monoids, free groups, trace monoids and their products.

Profinite metric on a monoid M of \mathcal{M}

Let, for each $(u, v) \in M^2$,

$$r(u, v) = \min \{ \text{Card}(N) \mid N \text{ separates } u \text{ and } v \}$$

$$d(u, v) = 2^{-r(u, v)}$$

with the usual conventions $\min \emptyset = +\infty$ and $2^{-\infty} = 0$.

Then

$$d(u, w) \leq \max(d(u, v), d(v, w)) \quad (\text{ultrametric})$$

$$d(uw, vw) \leq d(u, v)$$

$$d(wu, wv) \leq d(u, v)$$

Then (M, d) is a metric monoid.

Uniform continuity

Proposition

Let $M, N \in \mathcal{M}$. A function $f : M \rightarrow N$ is *preserves recognizability* if and only if it is *uniformly continuous* for the *profinite metrics*.

Uniform continuity

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Let $M, N \in \mathcal{M}$. A function $f : M \rightarrow N$ is *preserves recognizability* if and only if it is *uniformly continuous* for the *profinite metrics*.

What about *transductions* $\tau : M \rightarrow N$?

Hausdorff metric

The completion of the metric monoid (M, d) is a compact metric monoid \widehat{M} . The set $\mathcal{K}(\widehat{M})$ of compact subsets of \widehat{M} is also a compact monoid for the Hausdorff metric.

The Hausdorff metric on $\mathcal{K}(\widehat{M})$ is defined as follows. For $K, K' \in \mathcal{K}(\widehat{M})$, let

$$\delta(K, K') = \sup_{x \in K} d(x, K')$$

$$h(K, K') = \max(\delta(K, K'), \delta(K', K))$$

+ special definition if K or K' is empty

The case of transductions

Let M and N be monoids of \mathcal{M} and let $\tau : M \rightarrow N$ be a **transduction**.

Define a map $\hat{\tau} : M \rightarrow \mathcal{K}(\hat{N})$ by setting, for each $x \in M$, $\hat{\tau}(x) = \overline{\tau(x)}$.

Theorem (Pin-Silva 2005)

*The transduction τ is **preserves recognizability** iff $\hat{\tau}$ is **uniformly continuous** for the **Hausdorff metric**.*

An exercise

If L is regular, then

$$K = \{u \in A^* \mid u^{|u|} \in L\}$$

is also regular.

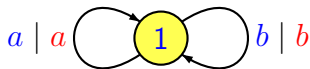
Proof. Indeed, $K = h^{-1}(L)$, where $h(u) = u^{|u|}$.

Now $h = g \circ f$

$$u \xrightarrow{f} (u, |u|) \xrightarrow{g} u^{|u|}$$

and f and g are both uniformly continuous. Thus K is regular.

Matrix representations



$$\mu(a) = a \quad \mu(b) = b \quad \mu(u) = u$$

$$f_1(u) = uu$$

$$f_1(u) = (\mu(u))^2$$

$$f_2(u) = uau^2$$

$$f_2(u) = \mu(u)a\mu(u)^2$$

$$\tau_1(u) = u^*$$

$$\tau_1(u) = \sum_{n \geq 0} \mu(u)^n$$

$$\tau_2(u) = \bigcup_{p \text{ prime}} u^p$$

$$\tau_2(u) = \sum_{p \text{ prime}} \mu(u)^p$$

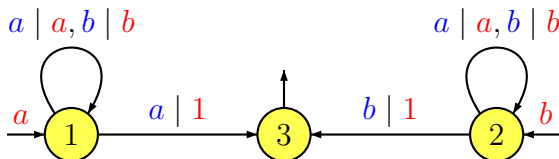
$$f(u) = a^{|u|_a} b^{|u|_b}$$



$$\mu(a) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \quad \mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \quad \mu(u) = \begin{pmatrix} a^{|u|_a} & 0 \\ 0 & b^{|u|_b} \end{pmatrix}$$

$$f(u) = \mu_{1,1}(u) \mu_{2,2}(u)$$

$$f(u) = \text{Last}(u)u$$



$$\mu(a) = \begin{pmatrix} a & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mu(b) = \begin{pmatrix} b & 0 & 0 \\ 0 & b & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mu(ua) = \begin{pmatrix} ua & 0 & u \\ 0 & ua & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mu(ub) = \begin{pmatrix} ub & 0 & 0 \\ 0 & ub & u \\ 0 & 0 & 0 \end{pmatrix}$$

$$f(u) = a\mu_{1,3}(u) + b\mu_{2,3}(u)$$

Matrix representations

A transduction $\tau: A^* \rightarrow M$ admits a **matrix representation** (S, μ) of **degree** n if there exist a monoid morphism $\mu: A^* \rightarrow \mathcal{P}(M)^{n \times n}$ and a possibly infinite union of products S involving **arbitrary subsets of** M and n^2 **variables**

$X_{1,1}, \dots, X_{n,n}$, such that, for all $u \in A^*$,

$$\tau(u) = S[\mu_{1,1}(u), \dots, \mu_{n,n}(u)].$$

Example for $n = 2$: Let $(P_k)_{k \geq 0}$ be subsets of M .

$$S = \bigcup_{k \in \mathbb{N}} P_0 X_{1,1}^k P_k X_{2,1} X_{1,1}^k X_{2,2} P_k X_{1,1} P_{2k}$$

Matrix representation of transducers

Theorem (Pin-Sakarovitch 1983)

Let (S, μ) be a *matrix representation* of *degree* n of a *transduction* $\tau: A^* \rightarrow M$. Let P be a subset of M recognised by a morphism $\eta: M \rightarrow N$. Then the language $\tau^{-1}(P)$ is *recognised* by the *submonoid* $\eta\mu(A^*)$ of the monoid of matrices $\mathcal{P}(N)^{n \times n}$.

Corollary

Every transduction having a *matrix representation* *preserves recognizability*.

An exercise

Let L be a regular language and $T \subseteq \mathbb{N}^2$. Then

$$L_T = \{u \in A^* \mid \text{there exist } x, y \text{ and } (p, q) \in T \\ \text{such that } |x| = p|u|, |y| = q|u| \text{ and } xuy \in L\}$$

is regular.

Observe that $L_T = \tau^{-1}(L)$ where the transduction $\tau(u) = \bigcup_{(p,q) \in T} A^{p|u|} u A^{q|u|}$ admits the matrix representation (S, μ) , with

$$\mu(u) = \begin{pmatrix} A^{|u|} & \emptyset & \emptyset \\ \emptyset & u & \emptyset \\ \emptyset & \emptyset & A^{|u|} \end{pmatrix} \text{ and } S = \bigcup_{(p,q) \in T} X_{1,1}^p X_{2,2} X_{3,3}^q$$

Some other examples

- **Intersection:** $L_1 \cap L_2 = \tau^{-1}(L_1 \times L_2)$ where $\tau : A^* \rightarrow A^* \times A^*$ is given by $\tau(u) = u \times u$.
- **Concatenation product:** $L_1 L_2 = \tau^{-1}(L_1 \times L_2)$ where $\tau(u) = \{(u_1, u_2) \mid u_1 u_2 = u\}$.
$$\mu(u) = \begin{pmatrix} (u, 1) & \{(u_1, u_2) \mid u = u_1 u_2\} \\ \emptyset & (1, u) \end{pmatrix}$$

(Schützenberger product)
- **Shuffle:** $L_1 \sqcup L_2 = \tau^{-1}(L_1 \times L_2)$ where $\tau(u) = \{(u_1, u_2) \mid u \in u_1 \sqcup u_2\}$. Here $\mu = \tau$.
- **Union, quotients, morphisms, inverses of morphisms, and many others**

Standard exercises

If L is regular, then so are $\sqrt{L} = \{u \mid uu \in L\}$ and $\frac{1}{2}(L) = \{\text{first halves of words in } L\}$. If L is star-free, then so is \sqrt{L} .

Proof. $\sqrt{L} = \tau^{-1}(L)$ where $\tau(u) = u^2$. Taking $\mu(u) = u$ shows that if a monoid recognizes L , then it also recognizes \sqrt{L} .

$\frac{1}{2}(L) = \tau^{-1}(L)$ where $\tau(u) = uA^{|u|}$. Take

$$\mu(u) = \begin{pmatrix} u & \emptyset \\ \emptyset & A^{|u|} \end{pmatrix}$$

Streaming string transducers

A **substitution** $\sigma : A^* \rightarrow B^*$ is a monoid morphism from A^* to $\mathcal{P}(B^*)$.

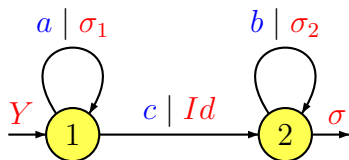
A **streaming string transducer** is a sequential transducer whose outputs are **substitutions**.

Theorem

Streaming string transducers preserve recognizability.

An example of streaming string transducer

The function $f(a^n c b^p) = a^p b^{pn}$ can be realized by the following streaming string transducer:

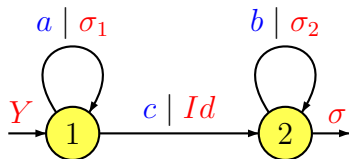


where $A = \{a, b, c\}$, $B = A \cup \{X, Y\}$ and $\sigma, \sigma_1, \sigma_2 : B^* \rightarrow B^*$ are substitutions defined by

$X\sigma_1 = X$	$Y\sigma_1 = YX$	$d\sigma_1 = d$ for $d \in A$
$X\sigma_2 = Xb$	$Y\sigma_2 = Ya$	$d\sigma_2 = d$ for $d \in A$
$X\sigma = 1$	$Y\sigma = 1$	$d\sigma = d$ for $d \in A$

Streaming string transducers at work

The function $f(a^n cb^p) = a^p b^{pn}$ can be realized by the following streaming string transducer:



$$\begin{aligned}\tau(a^n cb^p) &= Y \sigma_1^n \sigma_2^p \sigma = (Y X^n) \sigma_2^p \sigma = ((Y \sigma_2^p) (X \sigma_2^p)^n) \sigma \\ &= ((Y a^p) (X b^p)^n) \sigma = a^p b^{pn}\end{aligned}$$

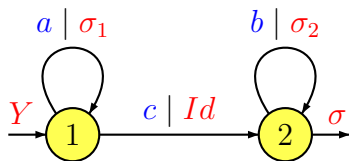
$$X \sigma_1 = X \quad Y \sigma_1 = YX \quad d \sigma_1 = d \text{ for } d \in A$$

$$X \sigma_2 = Xb \quad Y \sigma_2 = Ya \quad d \sigma_2 = d \text{ for } d \in A$$

$$X \sigma = 1 \quad Y \sigma = 1 \quad d \sigma = d \text{ for } d \in A$$

Matrix representation of a sst

The function $f(a^n cb^p) = a^p b^{pn}$ can be realized by the following **streaming string transducer**:



Let M be the **monoid** of all **substitutions** from B^* into itself under composition. Then

$\mu : A^* \rightarrow (M \cup \{0\})^{2 \times 2}$ is the morphism defined by

$$\mu(a) = \begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \mu(b) = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_2 \end{pmatrix} \quad \mu(c) = \begin{pmatrix} 0 & Id \\ 0 & 0 \end{pmatrix}$$

Effective computation

Let $R \in \text{Rec}(B^*)$ and let $\eta : B^* \rightarrow N$ be a monoid morphism recognizing R (N finite).

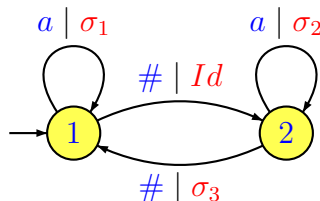
One can define a right action of M on the monoid $\mathcal{P}(N)^B$, which induces a monoid morphism π from M to the monoid T of all transformations on $\mathcal{P}(N)^B$.

Proposition (Pin, Reynier, Villevallois, 2018)

The language $\tau^{-1}(R)$ is recognized by the monoid morphism $\pi \circ \mu : A^ \rightarrow (T \cup \{0\})^{Q \times Q}$.*

An other example of streaming string transducer

$f(u_0\#u_1\#u_2\#u_3\#u_4\#u_5\#\cdots\#u_n) =$
 $u_1\#u_0\#u_3\#u_2\#u_5\#u_4\#\cdots\#u_n$ is realized by the sst:



where $A = \{a, b, \#\}$, $B = A \cup \{X, Y, Z\}$ and
 $\sigma, \sigma_1, \sigma_2 : B^* \rightarrow B^*$ are substitutions defined by

$$X\sigma_1 = X$$

$$Y\sigma_1 = Ya$$

$$Z\sigma_1 = Z$$

$$X\sigma_2 = X$$

$$Y\sigma_2 = Y$$

$$Z\sigma_2 = Za$$

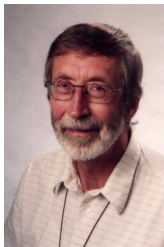
$$X\sigma_3 = XZ\#Y\#$$

$$Y\sigma_3 = 1$$

$$Z\sigma_3 = 1$$

Part VII

Functions from \mathbb{N} to \mathbb{N}



Siefkes,

Decidable extensions of monadic
second order successor arithmetic (1970)

Regularity-preserving functions from \mathbb{N} to \mathbb{N}

As we have seen, the **regularity-preserving functions** are exactly the **uniformly continuous functions** from \mathbb{N} to \mathbb{N} for the **profinite metric**.

A function $f: \mathbb{N} \rightarrow \mathbb{N}$ is **residually ultimately periodic** (rup) if, for each monoid morphism h from \mathbb{N} to a **finite** monoid, the sequence $h(f(n))$ is ultimately periodic.

Proposition

A function $f: \mathbb{N} \rightarrow \mathbb{N}$ is **uniformly continuous** iff it is **residually ultimately periodic**.

Ultimately periodic functions

A function $f : \mathbb{N} \rightarrow \mathbb{N}$ is **ultimately periodic** if there exists $t \geq 0$ and $p > 0$ such that, for all $n \geq t$, $f(n+p) = f(n)$. For instance, the sequence

$$1, 4, 0, 2, 8, 1, \underbrace{2, 3, 5}, \underbrace{2, 3, 5}, \underbrace{2, 3, 5}, \underbrace{2, 3, 5}, \dots$$

is ultimately periodic.

A function $f : \mathbb{N} \rightarrow \mathbb{N}$ is **ultimately periodic modulo n** if the function $f \bmod n$ is ultimately periodic. It is **cyclically ultimately periodic** (cup) if it is ultimately periodic modulo n for all $n > 0$.

Regularity-preserving functions from \mathbb{N} to \mathbb{N}

Theorem (Siefkes 1970, Seiferas-McNaughton 1976)

A function $f : \mathbb{N} \rightarrow \mathbb{N}$ is *ultimately periodic modulo n* iff for $0 \leq k < n$, the set $f^{-1}(k + n\mathbb{N})$ is *regular*.

Theorem (Siefkes 1970, Seiferas-McNaughton 1976)

A function $f : \mathbb{N} \rightarrow \mathbb{N}$ is *regularity-preserving* iff it is *cyclically ultimately periodic* and, for every $k \in \mathbb{N}$, the set $f^{-1}(k)$ is *regular*.

Regularity-preserving functions from \mathbb{N} to \mathbb{N}

[Siefkes 1970]

- Every polynomial function
- $n \rightarrow 2^n$
- $n \rightarrow n!$
- $n \rightarrow 2^{2^{2^{\dots^2}}}$ (exponential stack of 2's of height n)

[Carton-Thomas 02]

- $n \rightarrow F_n$ (Fibonacci number)
- $n \rightarrow t_n$, where t_n is the prefix of length n of the Prouhet-Thue-Morse sequence.

Counterexamples [Siefkes 1970]

- $n \rightarrow \lfloor \sqrt{n} \rfloor$ **is not** cyclically ultimately periodic and hence **not** regularity-preserving.
- $n \rightarrow \binom{2n}{n}$ **is not** ultimately periodic modulo 4 and hence **not** regularity-preserving. Indeed

$$\binom{2n}{n} \bmod 4 = \begin{cases} 2 & \text{if } n \text{ is a power of } 2, \\ 0 & \text{otherwise.} \end{cases}$$

Open problem?

- Is the function $n \rightarrow p_n$ regularity-preserving? (p_n is the n -th prime number).

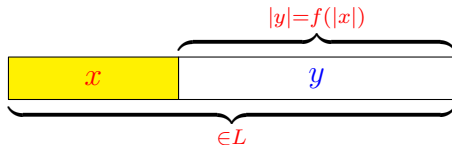
Application to languages

Theorem (Seiferas, McNaughton 1976)

Let $f: \mathbb{N} \rightarrow \mathbb{N}$ be a *rup function*. If L is *regular* [*star-free*], then the language

$$\left\{ x \mid \text{there exists some } y \text{ of length } f(|x|) \text{ such that } xy \in L \right\}$$

is also *regular* [*star-free*].



Subword filtering problem (A. B. Matos)

Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a **strictly increasing** function.

Filtering a word $u = a_0 a_1 \cdots a_n$ through f consists in just keeping the letters a_i such that i is in the range of f .

If L is **regular**, is the set of words of L **filtered** by f always **regular**?

Theorem (Berstel, Boasson, Carton, Petazzoni, P. (2006))

*This happens iff the function Δf defined by $\Delta f(n) = f(n+1) - f(n)$ is **regularity-preserving**.*

Connections with logic

A function $f : \mathbb{N} \rightarrow \mathbb{N}$ is **effectively regularity-preserving** if, for each given **regular** subset of \mathbb{N} , $f^{-1}(R)$ is **regular** and **effectively computable**.

Recall that $\Delta f(n) = f(n+1) - f(n)$.

Theorem (Carton-Thomas 02)

Let χ_P be the characteristic function of a **predicate** $P \subseteq \mathbb{N}$. If $\Delta\chi_P$ is **effectively regularity-preserving**, then the monadic second order theory $\text{MTh}(\mathbb{N}, <, P)$ is **decidable**.

Closure properties of cup functions

Theorem (Siefkes 70, Zhang 98, Carton-Thomas 02)

Let $f, g : \mathbb{N} \rightarrow \mathbb{N}$ be *cyclically ultimately periodic* functions. Then so are the following functions:

- (1) $g \circ f$, $f + g$, fg , f^g , and $f - g$ provided that $f \geq g$ and $\lim_{n \rightarrow \infty} (f - g)(n) = +\infty$,
- (2) (*generalised sum*) $n \rightarrow \sum_{0 \leq i \leq g(n)} f(i)$,
- (3) (*generalised product*) $n \rightarrow \prod_{0 \leq i \leq g(n)} f(i)$.

Closure properties of rup functions

Theorem

Let f and g be *rup functions*. Then so are $f \circ g$, $f + g$, fg , f^g , $\sum_{0 \leq i \leq g(n)} f(i)$, $\prod_{0 \leq i \leq g(n)} f(i)$.

Let $f : \mathbb{N} \rightarrow \{0, 1\}$ be a *non-recursive* function. Then the function $n \rightarrow (\sum_{0 \leq i \leq n} f(i))!$ is *regularity-preserving* but *non-recursive*.

Open question. Is it possible to describe the *primitive recursive cup [rup] functions*? One could try to use a recursion scheme similar to *Siefkes' primitive recursion scheme* for cup functions.

Siefkes' recursion scheme (1970)

Theorem

Let $g : \mathbb{N}^k \rightarrow \mathbb{N}$ and $h : \mathbb{N}^{k+2} \rightarrow \mathbb{N}$ be *cyclically ultimately periodic* functions satisfying three technical conditions. Then the function f defined from g and h by *primitive recursion*, i.e.

$$\begin{aligned} f(0, x_1, \dots, x_k) &= g(x_1, \dots, x_k), \\ f(n+1, x_1, \dots, x_k) &= \\ &\quad h(n, x_1, \dots, x_k, f(n, x_1, \dots, x_k)) \end{aligned}$$

is *cyclically ultimately periodic*.

The three technical conditions

- (1) h is cyclically ultimately periodic in x_{k+2} of decreasing period,
- (2) g is essentially increasing in x_k ,
- (3) for all $x \in \mathbb{N}^{k+2}$, $x_{k+2} < h(x_1, \dots, x_{k+2})$.

A function f is essentially increasing in x_j iff, for all $z \in \mathbb{N}$, there exists $y \in \mathbb{N}$ such that for all $x \in \mathbb{N}^n$, $y \leq x_j$ implies $z \leq f(x_1, \dots, x_n)$.

A function f is c.u.p. of decreasing period in x_j iff, for all p , the period of the function $f \bmod p$ in x_j is $\leq p$.