

Derived terms without derivation

A shifted perspective on the derived-term automaton

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CNRS / Université Paris Cité and Télécom Paris, IPP

Joint work with

Sylvain Lombardy

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Derived terms without derivation

Truth emerges from generalisation

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The results presented in this talk are based
on a work that has been published recently in the
Journal of Computer Science and Cybernetics **37** (2021)
by the Vietnamese Academy of Science and Technology.

It can also be found at <https://arxiv.org/abs/2110.09181>

In the beginning, Kleene created the family of regular events and the family of recognizable events.

He established that these two families coincide.

He looked at it and thought it was good, and moved on to something else.

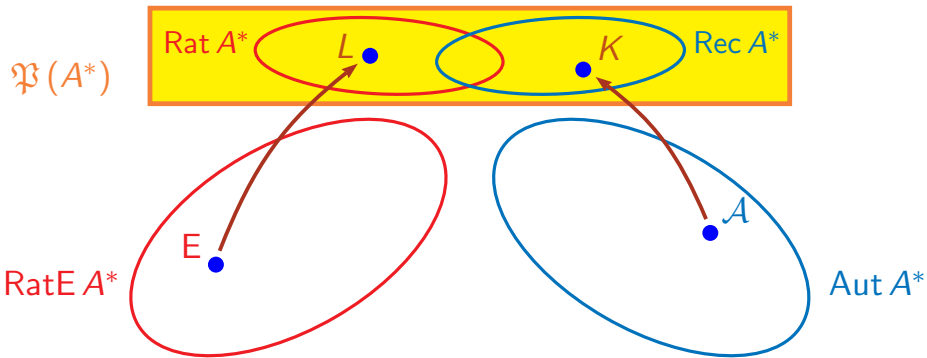
$$\text{Rat } A^* = \text{Rec } A^*$$

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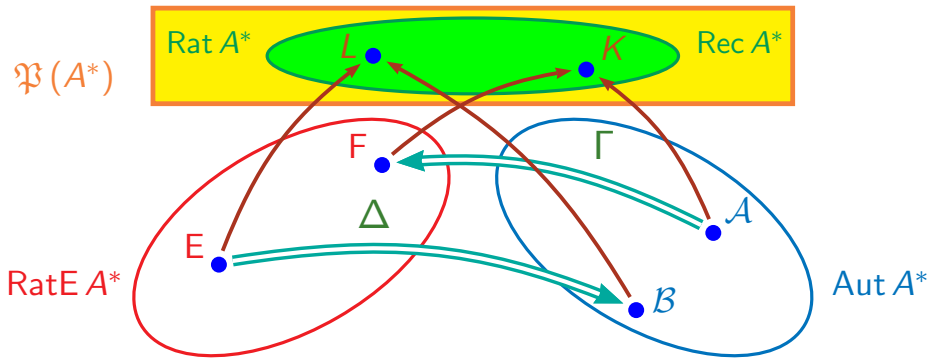
Notations

- ▶ A *alphabet*, i.e. a finite set of *letters*
- ▶ A^* set of *words*
- ▶ $L \subseteq A^*$ *language*
- ▶ $\text{Rat } A^*$ set of *regular languages* over A^*
 $\text{RatE } A^*$ set of *regular expressions* over A^*
 $|E|$ language *denoted* by the expression E
- ▶ $\text{Rec } A^*$ set of *recognizable languages* over A^*
 $\text{Aut } A^*$ set of *finite automata* over A^*
 $|\mathcal{A}|$ language *accepted* by the automaton \mathcal{A}

$$\text{Rat } A^* = \text{Rec } A^*$$

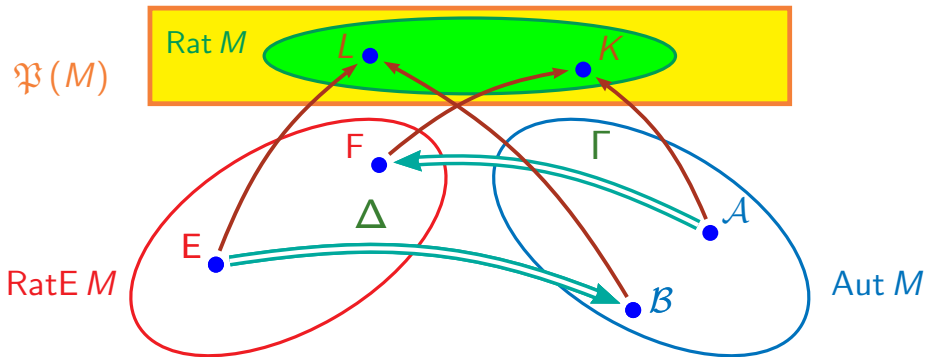


$$\text{Rat } A^* = \text{Rec } A^*$$

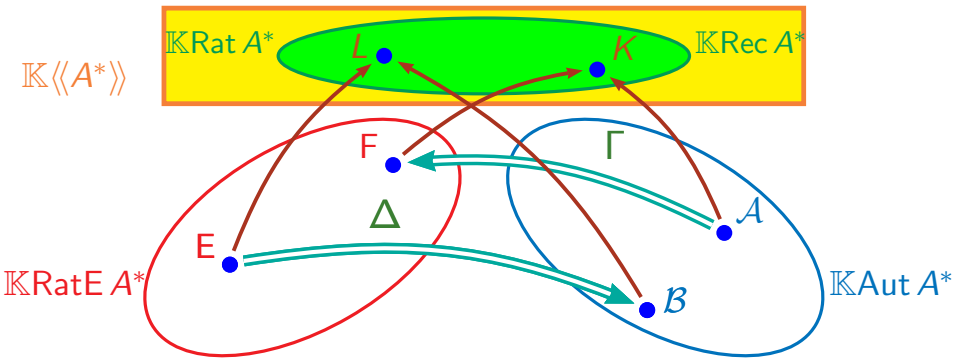


M non free monoid

$\text{Rat } M \neq \text{Rec } M$

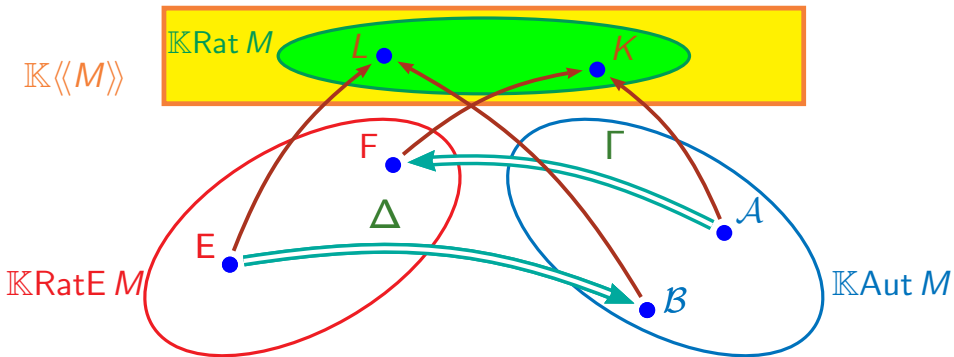


$$\mathbb{K}\text{Rat } A^* = \mathbb{K}\text{Rec } A^*$$

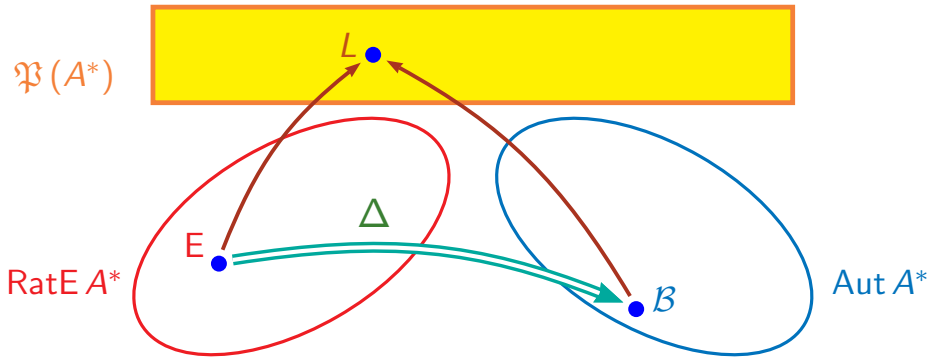


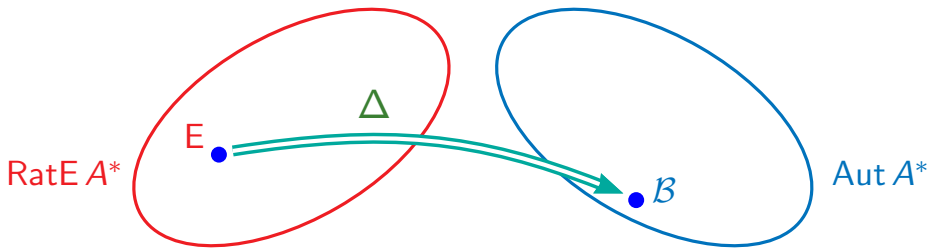
M non free monoid

$\mathbb{K}\text{Rat } M \neq \mathbb{K}\text{Rec } M$



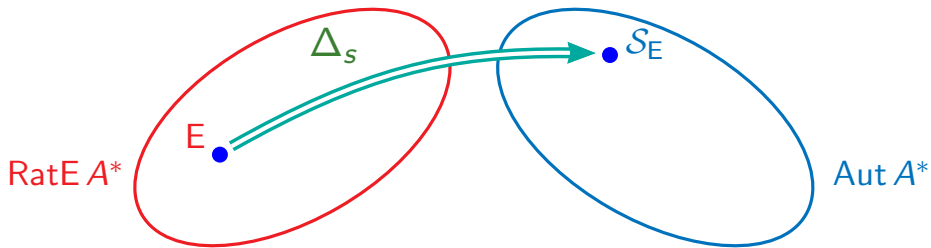
$$\text{Rat } A^* = \text{Rec } A^*$$





► Standard automaton of E

position, Glushkov

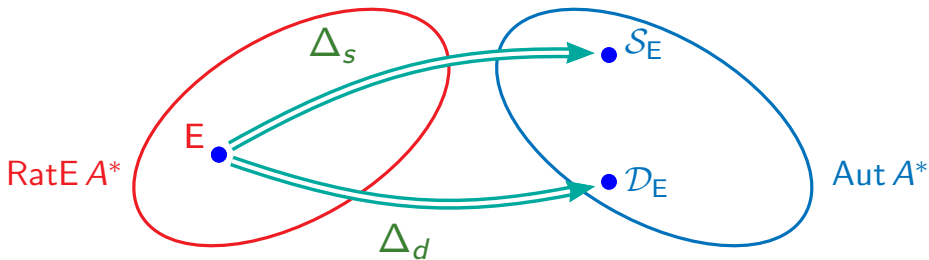


Th. [Glushkov 61]

$$|\mathcal{S}_E| = |E|$$

$$\dim \mathcal{S}_E = \ell(E) + 1$$

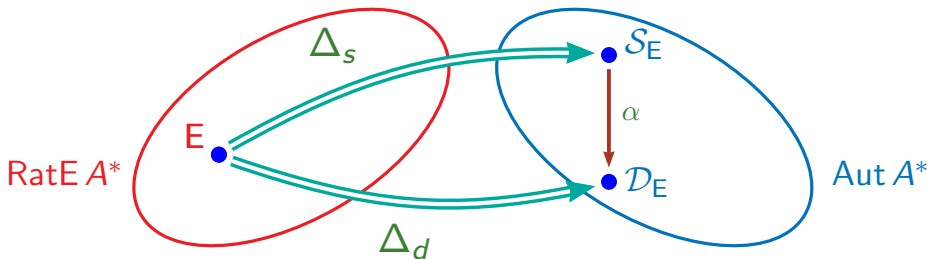
- ▶ Standard automaton of E position, Glushkov
- ▶ Derived term automaton of E Brzozowski–Antimirov



Th. [Glushkov 61] $|\mathcal{S}_E| = |E|$ $\dim \mathcal{S}_E = \ell(E) + 1$

Th. [Brzozowski–Antimirov 64–96] $|\mathcal{D}_E| = |E|$ $\dim \mathcal{D}_E \leq \ell(E) + 1$

- ▶ Standard automaton of E position, Glushkov
- ▶ Derived term automaton of E Brzozowski–Antimirov

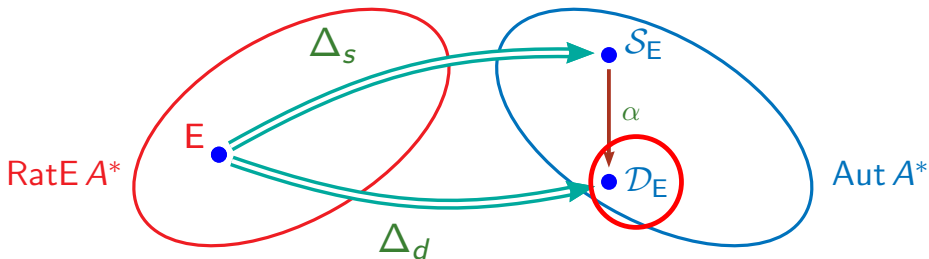


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Th. [Champarnaud-Ziadi 02] $\alpha: \mathcal{S}_E \rightarrow \mathcal{D}_E$ morphism of automata.

- ▶ Standard automaton of E position, Glushkov
- ▶ Derived term automaton of E Brzozowski–Antimirov



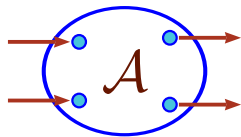
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Th. [Champarnaud-Ziadi 02] $\alpha: \mathcal{S}_E \rightarrow \mathcal{D}_E$ morphism of automata.

The standard automaton of an expression

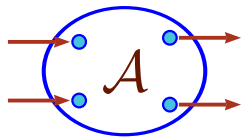
Definition of an automaton



$$\mathcal{A} = \langle I, E, T \rangle$$

The standard automaton of an expression

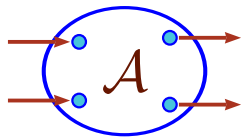
Definition of an automaton



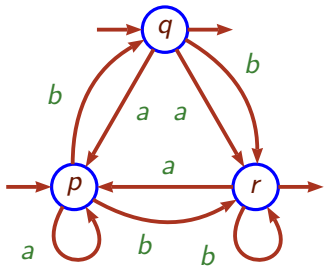
$$\mathcal{A} = \left\langle \left(\boxed{I} \right), \left(\boxed{E} \right), \left(\boxed{T} \right) \right\rangle$$

The standard automaton of an expression

Definition of an automaton



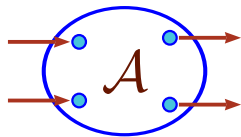
$$\mathcal{A} = \left\langle \left(\boxed{I} \right), \left(\boxed{E} \right), \left(\boxed{T} \right) \right\rangle$$



$$\mathcal{A} = \left\langle \left(1 \ 1 \ 0 \right), \begin{pmatrix} a & b & b \\ a & 0 & a+b \\ a & 0 & b \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

The standard automaton of an expression

Definition of an automaton



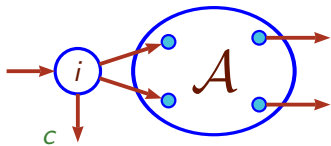
$$\mathcal{A} = \left\langle \left(\boxed{I} \right), \left(\boxed{E} \right), \left(\boxed{T} \right) \right\rangle$$

The language accepted by \mathcal{A}

$$|\mathcal{A}| = I \cdot E^* \cdot T$$

The standard automaton of an expression

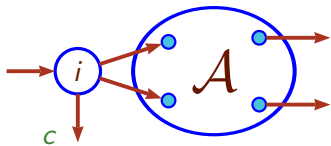
Definition of a standard automaton



$$\mathcal{A} = \left\langle \left(\begin{array}{|c|} \hline 1 & 0 \\ \hline \end{array} \right), \left(\begin{array}{|c|c|} \hline 0 & J \\ \hline 0 & F \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline c \\ \hline U \\ \hline \end{array} \right) \right\rangle$$

The standard automaton of an expression

Definition of a standard automaton



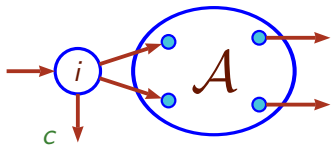
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The language accepted by \mathcal{A}

$$|\mathcal{A}| = c + J \cdot F^* \cdot U$$

The standard automaton of an expression

Definition of a standard automaton



$$\mathcal{A} = \left\langle \left(\begin{array}{|c|} \hline 1 & 0 \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline 0 & J \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline c \\ \hline \end{array} \right) \right\rangle$$

Operations on standard automata

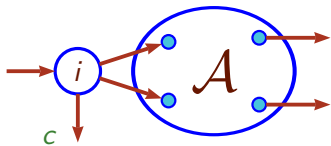
$$\mathcal{A} + \mathcal{B}$$

$$\mathcal{A} \cdot \mathcal{B}$$

$$\mathcal{A}^*$$

The standard automaton of an expression

Definition of a standard automaton



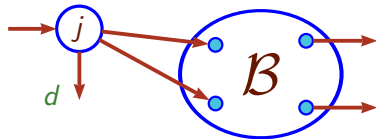
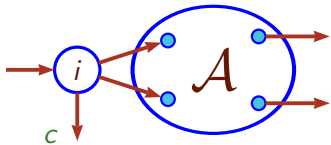
$$\mathcal{A} = \left\langle \left(1 \begin{array}{|c|} \hline 0 \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline 0 \begin{array}{|c|} \hline J \\ \hline \end{array} \\ \hline 0 \begin{array}{|c|} \hline F \\ \hline \end{array} \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline c \\ \hline U \\ \hline \end{array} \right) \right\rangle$$

Operations on standard automata

$\mathcal{A} + \mathcal{B}$

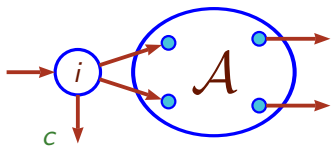
$\mathcal{A} \cdot \mathcal{B}$

\mathcal{A}^*



The standard automaton of an expression

Definition of a standard automaton



$$\mathcal{A} = \left\langle \left(1 \begin{array}{|c|} \hline 0 \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline 0 \begin{array}{|c|} \hline J \\ \hline \end{array} \\ \hline 0 \begin{array}{|c|} \hline F \\ \hline \end{array} \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline c \\ \hline U \\ \hline \end{array} \right) \right\rangle$$

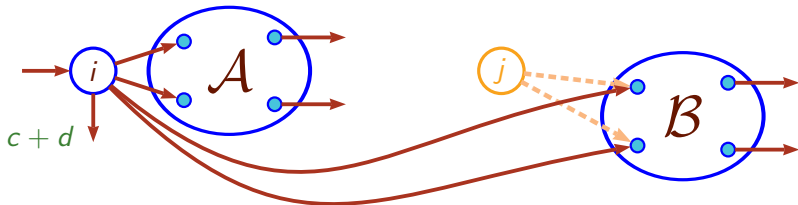
Operations on standard automata

$$\mathcal{A} + \mathcal{B}$$

$$\mathcal{A} \cdot \mathcal{B}$$

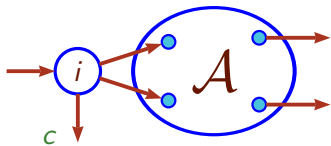
$$\mathcal{A}^*$$

$$|\mathcal{A} + \mathcal{B}| = |\mathcal{A}| \cup |\mathcal{B}|$$



The standard automaton of an expression

Definition of a standard automaton



$$\mathcal{A} = \left\langle \left(1 \begin{array}{|c|} \hline 0 \\ \hline \end{array} \right), \left(\begin{array}{|c|c|} \hline 0 & J \\ \hline 0 & F \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline c \\ \hline U \\ \hline \end{array} \right) \right\rangle$$

Operations on standard automata

$$\mathcal{A} + \mathcal{B}$$

$$\mathcal{A} \cdot \mathcal{B}$$

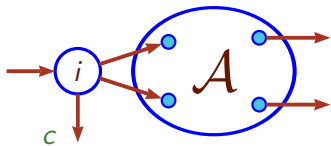
$$\mathcal{A}^*$$

$$|\mathcal{A} + \mathcal{B}| = |\mathcal{A}| \cup |\mathcal{B}|$$

$$\mathcal{A} + \mathcal{B} = \left\langle \left(1 \begin{array}{|c|c|} \hline 0 & 0 \\ \hline \end{array} \right), \left(\begin{array}{|c|c|c|} \hline 0 & J & K \\ \hline 0 & F & 0 \\ \hline 0 & 0 & G \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline c+d \\ \hline U \\ \hline V \\ \hline \end{array} \right) \right\rangle$$

The standard automaton of an expression

Definition of a standard automaton



$$\mathcal{A} = \left\langle \left(1 \begin{array}{|c|} \hline 0 \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline 0 \begin{array}{|c|} \hline J \\ \hline \end{array} \\ \hline 0 \begin{array}{|c|} \hline F \\ \hline \end{array} \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline c \\ \hline U \\ \hline \end{array} \right) \right\rangle$$

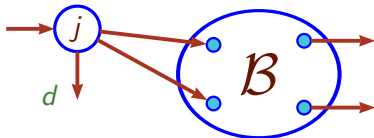
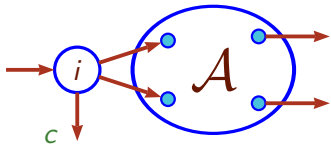
Operations on standard automata

$$\mathcal{A} + \mathcal{B}$$

$$|\mathcal{A} + \mathcal{B}| = |\mathcal{A} \cup \mathcal{B}|$$

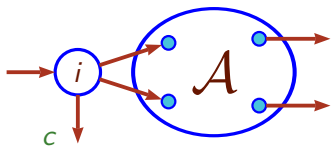
$$\mathcal{A} \cdot \mathcal{B}$$

$$\mathcal{A}^*$$



The standard automaton of an expression

Definition of a standard automaton



$$\mathcal{A} = \left\langle \left(1 \begin{array}{|c|} \hline 0 \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline 0 \begin{array}{|c|} \hline J \\ \hline \end{array} \\ \hline 0 \begin{array}{|c|} \hline F \\ \hline \end{array} \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline c \\ \hline U \\ \hline \end{array} \right) \right\rangle$$

Operations on standard automata

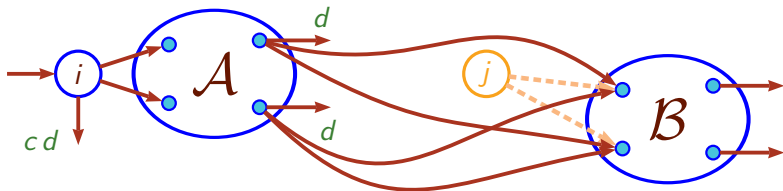
$$\mathcal{A} + \mathcal{B}$$

$$|\mathcal{A} + \mathcal{B}| = |\mathcal{A}| \cup |\mathcal{B}|$$

$$\mathcal{A} \cdot \mathcal{B}$$

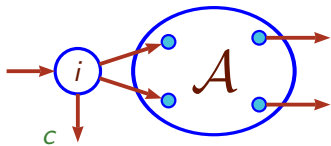
$$|\mathcal{A} \cdot \mathcal{B}| = |\mathcal{A}| \cdot |\mathcal{B}|$$

$$\mathcal{A}^*$$



The standard automaton of an expression

Definition of a standard automaton



$$\mathcal{A} = \left\langle \left(1 \begin{array}{|c|} \hline 0 \\ \hline \end{array} \right), \left(\begin{array}{|c|c|} \hline 0 & J \\ \hline 0 & F \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline c \\ \hline U \\ \hline \end{array} \right) \right\rangle$$

Operations on standard automata

$$\mathcal{A} + \mathcal{B}$$

$$\mathcal{A} \cdot \mathcal{B}$$

$$\mathcal{A}^*$$

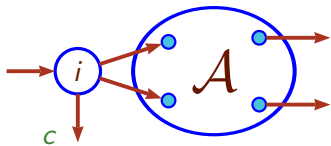
$$|\mathcal{A} + \mathcal{B}| = |\mathcal{A}| \cup |\mathcal{B}|$$

$$|\mathcal{A} \cdot \mathcal{B}| = |\mathcal{A}| \cdot |\mathcal{B}|$$

$$\mathcal{A} \cdot \mathcal{B} = \left\langle \left(1 \begin{array}{|c|c|} \hline 0 & 0 \\ \hline \end{array} \right), \left(\begin{array}{|c|c|c|} \hline 0 & J & cK \\ \hline 0 & F & U \cdot K \\ \hline 0 & 0 & G \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline cd \\ \hline Ud \\ \hline V \\ \hline \end{array} \right) \right\rangle$$

The standard automaton of an expression

Definition of a standard automaton



$$\mathcal{A} = \left\langle \left(\begin{array}{|c|} \hline 1 & 0 \\ \hline \end{array} \right), \left(\begin{array}{|c|c|} \hline 0 & J \\ \hline 0 & F \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline c \\ \hline U \\ \hline \end{array} \right) \right\rangle$$

Operations on standard automata

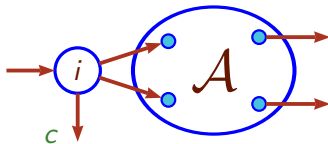
$$\mathcal{A} + \mathcal{B}$$

$$|\mathcal{A} + \mathcal{B}| = |\mathcal{A}| \cup |\mathcal{B}|$$

$$\mathcal{A} \cdot \mathcal{B}$$

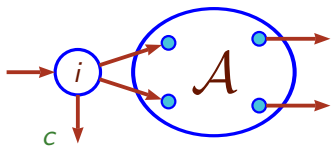
$$|\mathcal{A} \cdot \mathcal{B}| = |\mathcal{A}| \cdot |\mathcal{B}|$$

$$\mathcal{A}^*$$



The standard automaton of an expression

Definition of a standard automaton



$$\mathcal{A} = \left\langle \left(\begin{array}{|c|} \hline 1 \\ \hline 0 \end{array} \right), \left(\begin{array}{|c|} \hline 0 & J \\ \hline 0 & F \end{array} \right), \left(\begin{array}{|c|} \hline c \\ \hline U \end{array} \right) \right\rangle$$

Operations on standard automata

$$\mathcal{A} + \mathcal{B}$$

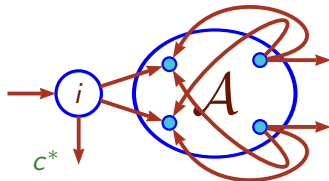
$$|\mathcal{A} + \mathcal{B}| = |\mathcal{A}| \cup |\mathcal{B}|$$

$$\mathcal{A} \cdot \mathcal{B}$$

$$|\mathcal{A} \cdot \mathcal{B}| = |\mathcal{A}| \cdot |\mathcal{B}|$$

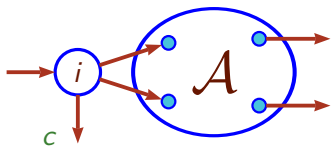
$$\mathcal{A}^*$$

$$|\mathcal{A}^*| = |\mathcal{A}|^*$$



The standard automaton of an expression

Definition of a standard automaton



$$\mathcal{A} = \left\langle \left(\begin{array}{|c|} \hline 1 & 0 \\ \hline \end{array} \right), \left(\begin{array}{|c|c|} \hline 0 & J \\ \hline 0 & F \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline c \\ \hline U \\ \hline \end{array} \right) \right\rangle$$

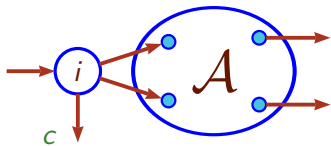
Operations on standard automata

$$\begin{array}{ccc} \mathcal{A} + \mathcal{B} & \mathcal{A} \cdot \mathcal{B} & \mathcal{A}^* \\ |\mathcal{A} + \mathcal{B}| = |\mathcal{A}| \cup |\mathcal{B}| & |\mathcal{A} \cdot \mathcal{B}| = |\mathcal{A}| \cdot |\mathcal{B}| & |\mathcal{A}^*| = |\mathcal{A}|^* \end{array}$$

$$\mathcal{A}^* = \left\langle \left(\begin{array}{|c|} \hline 1 & 0 \\ \hline \end{array} \right), \left(\begin{array}{|c|c|} \hline 0 & J \\ \hline 0 & H \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline 1 \\ \hline U \\ \hline \end{array} \right) \right\rangle \quad \text{with } H = U \cdot J + F$$

The standard automaton of an expression

Definition of a standard automaton



$$\mathcal{A} = \left\langle \left(\begin{array}{|c|} \hline 1 & 0 \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline 0 & J \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline c \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline 0 & F \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline U \\ \hline \end{array} \right) \right\rangle$$

Operations on standard automata

$$\mathcal{A} + \mathcal{B}$$

$$|\mathcal{A} + \mathcal{B}| = |\mathcal{A}| \cup |\mathcal{B}|$$

$$\mathcal{A} \cdot \mathcal{B}$$

$$|\mathcal{A} \cdot \mathcal{B}| = |\mathcal{A}| \cdot |\mathcal{B}|$$

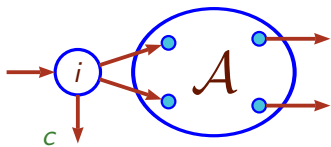
$$\mathcal{A}^*$$

$$|\mathcal{A}^*| = |\mathcal{A}|^*$$

Definition of \mathcal{S}_E

The standard automaton of an expression

Definition of a standard automaton



$$\mathcal{A} = \left\langle \left(1 \begin{array}{|c|} \hline 0 \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline J \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline c \\ \hline U \\ \hline \end{array} \right) \right\rangle$$

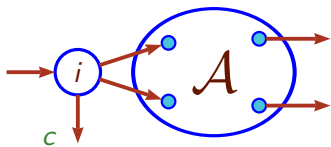
Definition of \mathcal{S}_E

Start from

$$\mathcal{S}_0 = \begin{array}{c} \rightarrow \\ \circ \\ i \end{array}, \mathcal{S}_1 = \begin{array}{c} \rightarrow \\ \circ \\ i \end{array} \rightarrow, \text{ and } \mathcal{S}_a = \begin{array}{c} \rightarrow \\ \circ \\ i \end{array} \xrightarrow{a} \begin{array}{c} \rightarrow \\ \circ \end{array}$$

The standard automaton of an expression

Definition of a standard automaton



$$\mathcal{A} = \left\langle \left(\begin{array}{|c|} \hline 1 & 0 \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline 0 & J \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline c \\ \hline \end{array} \right) \right\rangle$$

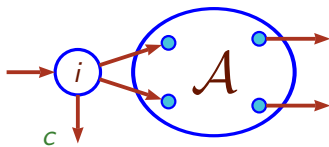
Definition of \mathcal{S}_E

Start from

$$\mathcal{S}_0 = \langle (1), (0), (0) \rangle, \mathcal{S}_1 = \langle (1), (0), (1) \rangle, \text{ and } \mathcal{S}_a = \left\langle \left(\begin{array}{|c|} \hline 1 & 0 \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline 0 & a \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline 0 \\ \hline 1 \\ \hline \end{array} \right) \right\rangle$$

The standard automaton of an expression

Definition of a standard automaton



$$\mathcal{A} = \left\langle \left(\begin{array}{|c|} \hline 1 & 0 \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline 0 & J \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline c \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline 0 & F \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline U \\ \hline \end{array} \right) \right\rangle$$

Definition of \mathcal{S}_E

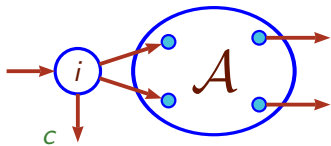
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Define $\mathcal{S}_{F+G} = \mathcal{S}_F + \mathcal{S}_G$, $\mathcal{S}_{F \cdot G} = \mathcal{S}_F \cdot \mathcal{S}_G$, and $\mathcal{S}_{F^*} = (\mathcal{S}_F)^*$

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Definition of \mathcal{S}_E

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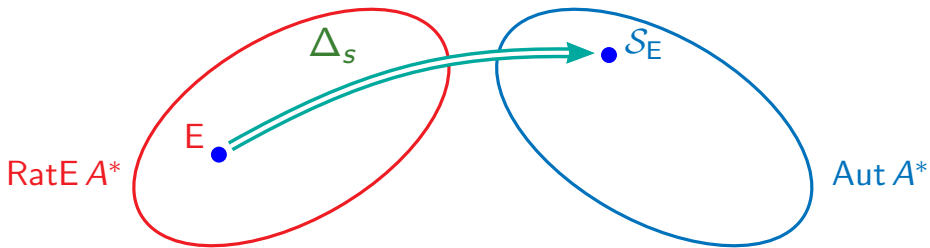
Define $\mathcal{S}_{F+G} = \mathcal{S}_F + \mathcal{S}_G$, $\mathcal{S}_{F \cdot G} = \mathcal{S}_F \cdot \mathcal{S}_G$, and $\mathcal{S}_{F^*} = (\mathcal{S}_F)^*$

And get $\mathcal{S}_E = \Delta_s(E)$

▶ Standard automaton of E

position, Glushkov

▶



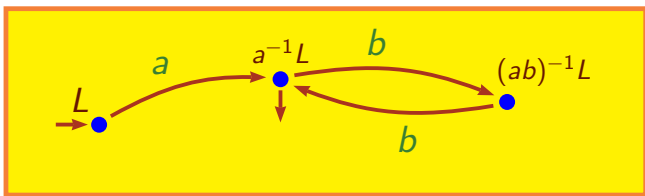
Th. [Glushkov 61]

$$|\mathcal{S}_E| = |E|$$

$$\dim \mathcal{S}_E = \ell(E) + 1$$

The Brzozowski derivative automaton

$\mathfrak{P}(A^*)$



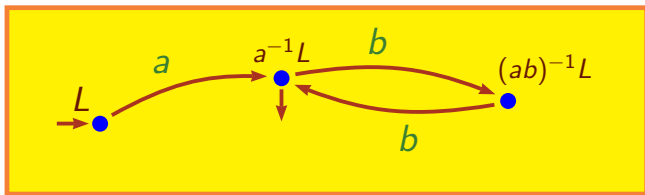
Preliminary

- ▶ **Th.** A language L is *recognisable* iff it has a *finite* number of (left) *quotients*.
- ▶ Construction of \mathcal{A}_L , the *minimal (deterministic) automaton* of L by means of the *quotients* of L

$$u^{-1}L = \{v \in A^* \mid uv \in L\}$$

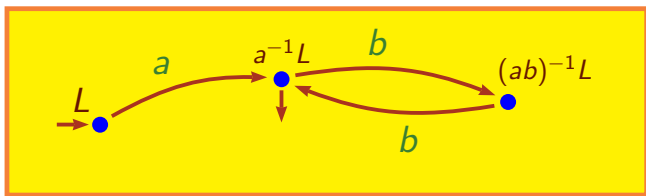
The Brzozowski derivative automaton

$\mathfrak{B}(A^*)$



The Brzozowski derivative automaton

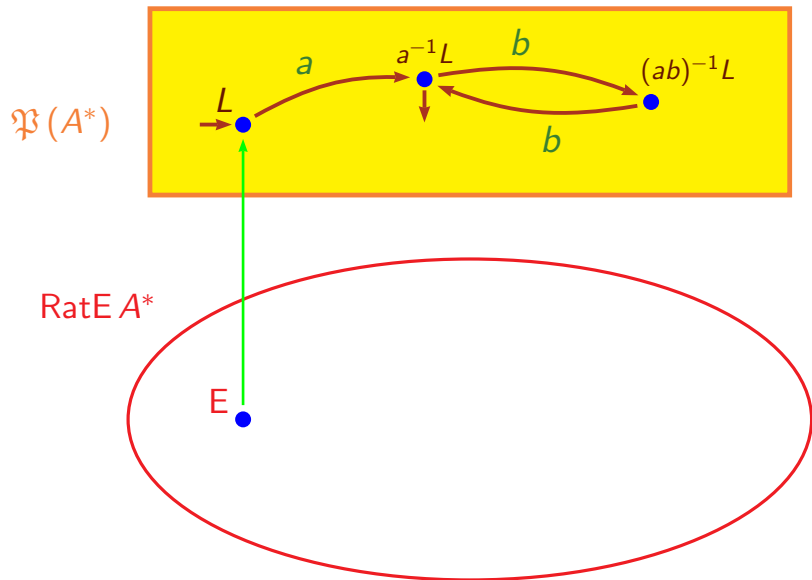
$\mathfrak{P}(A^*)$



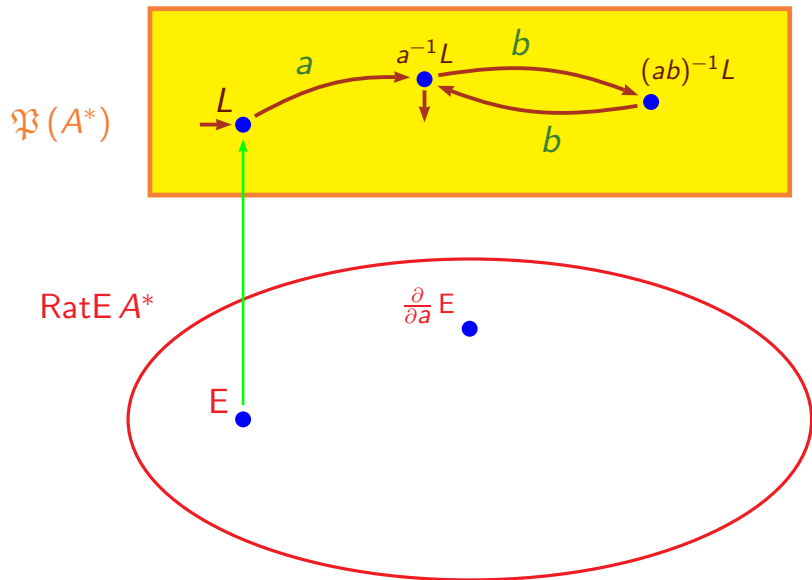
$\text{RatE } A^*$



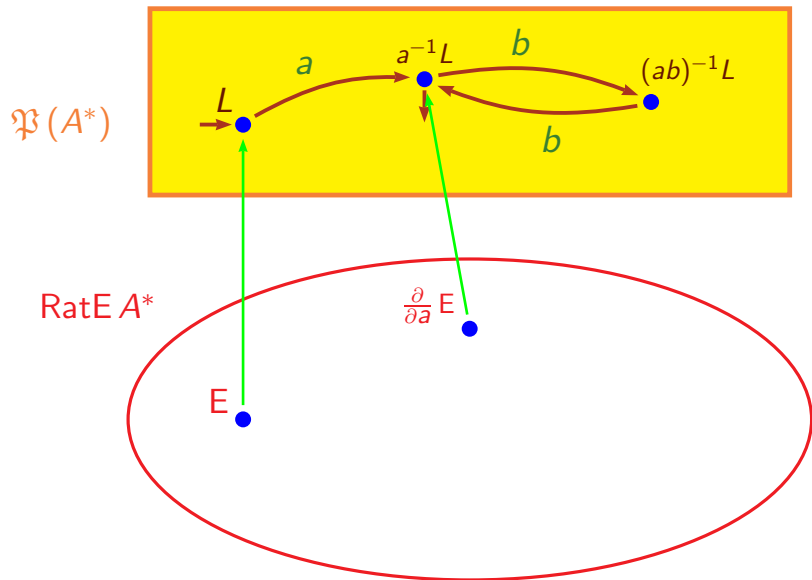
The Brzozowski derivative automaton



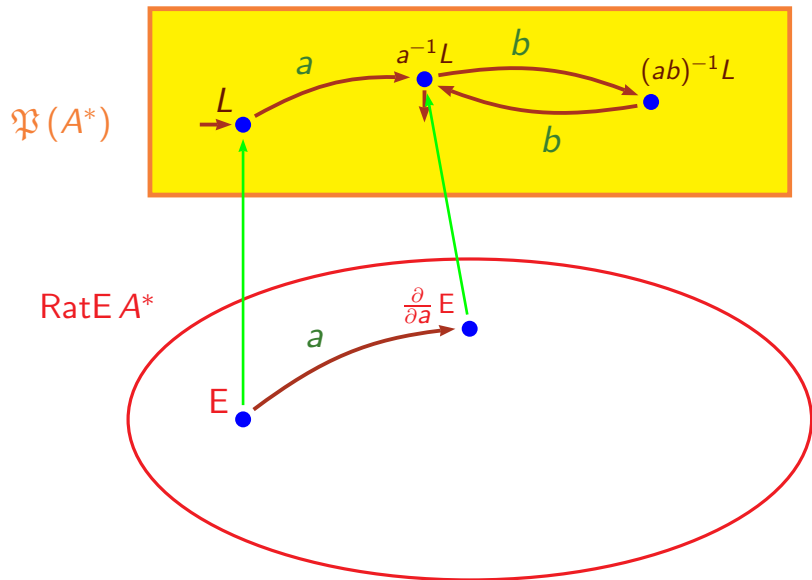
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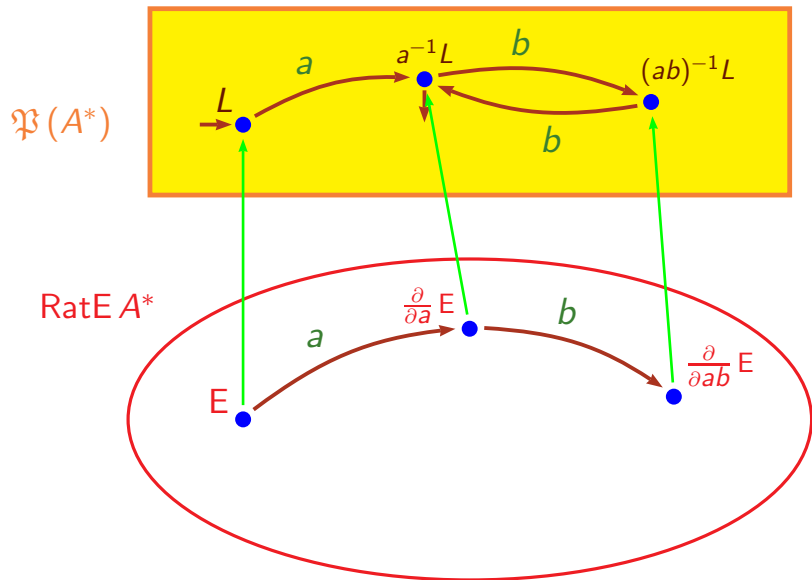
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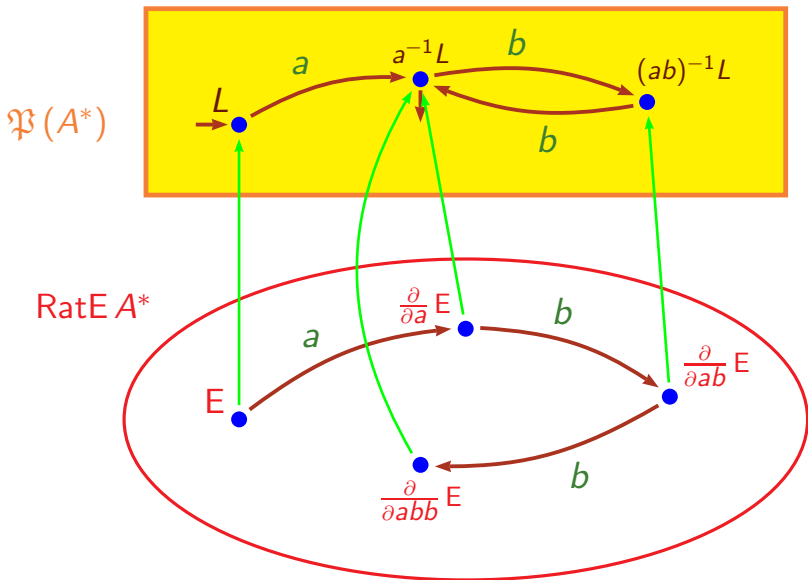
The Brzozowski derivative automaton



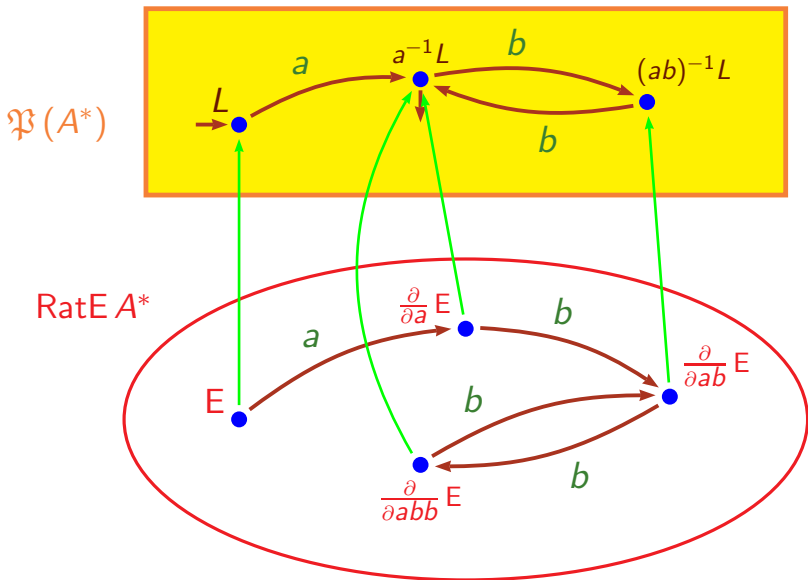
The Brzozowski derivative automaton



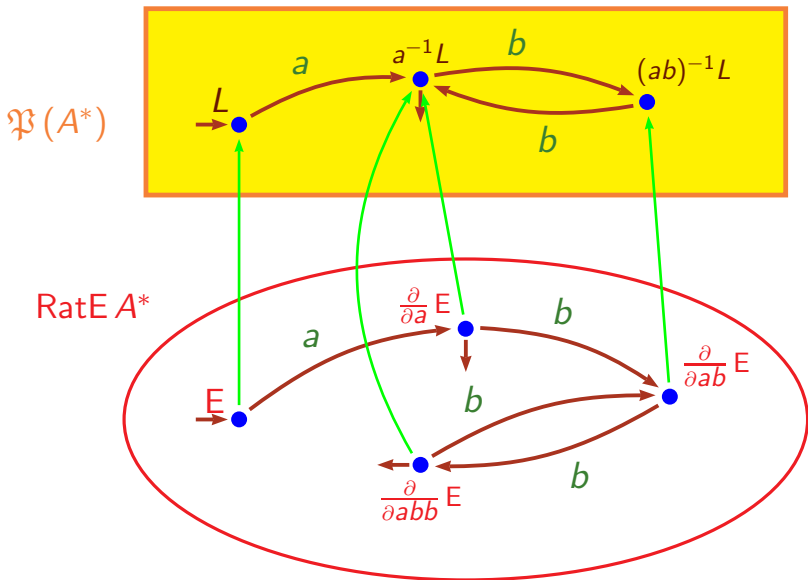
The Brzozowski derivative automaton



The Brzozowski derivative automaton



The Brzozowski derivative automaton



The Brzozowski derivation

Definition (Brzozowski 64)

$E \in \text{Rat}EA^*$ $\frac{\partial}{\partial a} E$ by induction on E

$$\frac{\partial}{\partial a} 0 = \frac{\partial}{\partial a} 1 = 0, \quad \frac{\partial}{\partial a} b = \begin{cases} 1 & \text{if } b = a \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial}{\partial a} (F + G) = \frac{\partial}{\partial a} F + \frac{\partial}{\partial a} G$$

$$\frac{\partial}{\partial a} (F \cdot G) = \left[\frac{\partial}{\partial a} F \right] \cdot G + c(F) \frac{\partial}{\partial a} G$$

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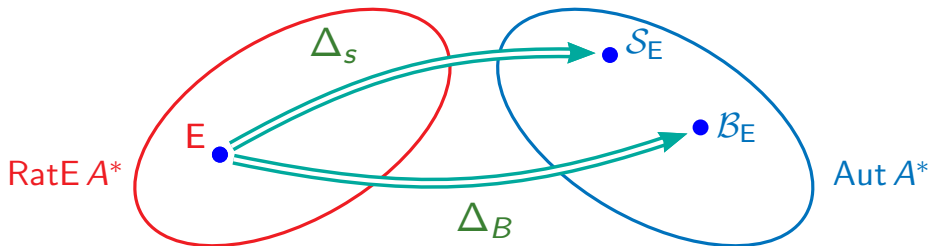
Theorem (Brzozowski 64)

For every E , there is a *finite* number of *derivatives*
modulo **A**, **C**, and **I**.

- ▶ Standard automaton of E
- ▶ Derivative automaton of E

position, Glushkov

Brzozowski



Th. [Glushkov 61]

$$|S_E| = |E|$$

$$\dim S_E = \ell(E) + 1$$

Th. [Brzozowski 64] $|B_E| = |E|$.

The Brzowski–Antimirov derivation

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The Brzozowski–Antimirov derivation

Definition (Brzozowski 64 — Antimirov 96)

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$$\frac{\partial}{\partial a} \left[\bigcup_{i \in I} F_i \right] = \bigcup_{i \in I} \frac{\partial}{\partial a} F_i, \quad \left[\bigcup_{i \in I} F_i \right] \cdot G = \bigcup_{i \in I} (F_i \cdot G).$$

The Brzowski–Antimirov derivation

Definition (Brzowski 64 — Antimirov 96)

$E \in \text{Rat}EA^*$ $\frac{\partial}{\partial a} E$ by induction on E $\frac{\partial}{\partial ua} E = \frac{\partial}{\partial a} \left(\frac{\partial}{\partial u} E \right)$

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Theorem (Antimirov 96)

For every E , there is a *finite* number of *derived terms*

The Brzowski–Antimirov derivation

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$$E \in \text{Rat}EA^* \quad \frac{\partial}{\partial a} E \text{ by induction on } E \quad \frac{\partial}{\partial ua} E = \frac{\partial}{\partial a} \left(\frac{\partial}{\partial u} E \right)$$

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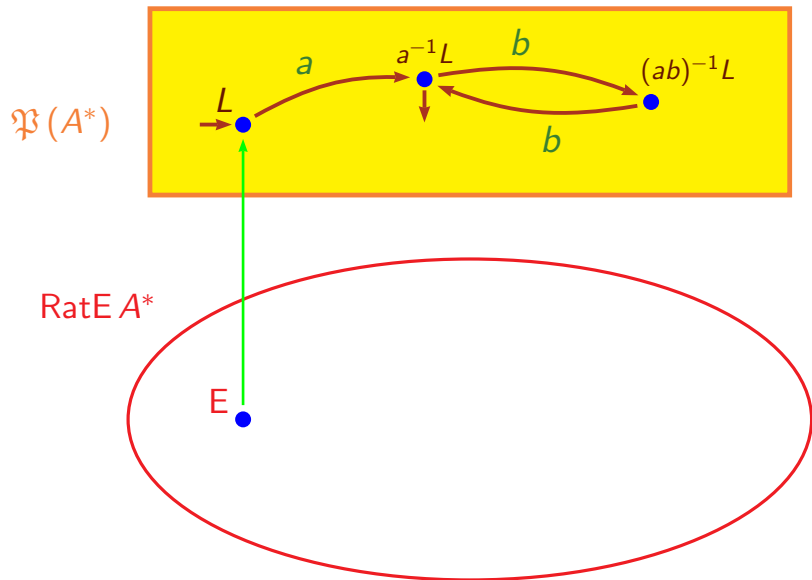
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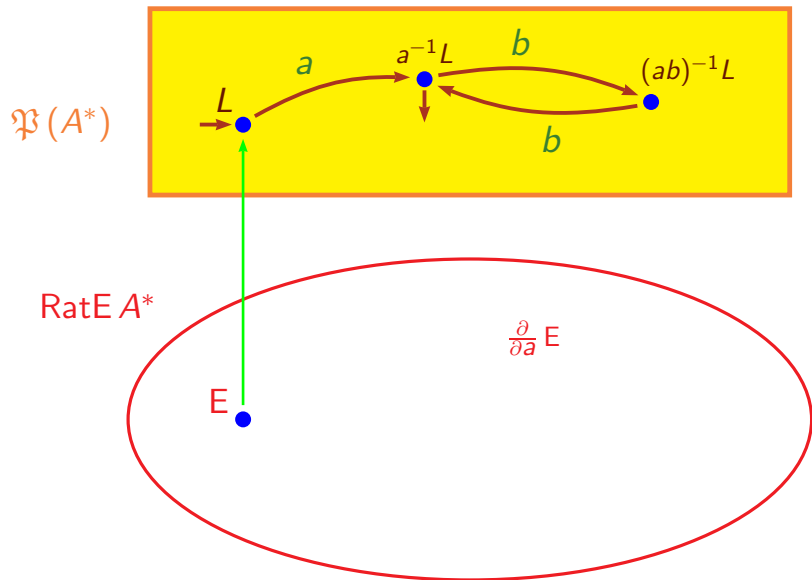
Theorem (Antimirov 96)

For every E , there is a *finite* number of *derived terms*
($\leq \ell(E) + 1$)

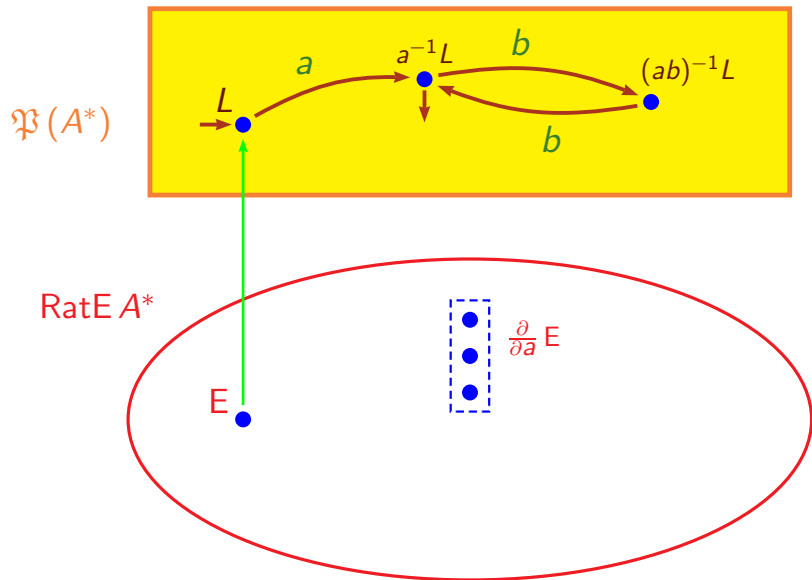
The Antimirov derived term automaton



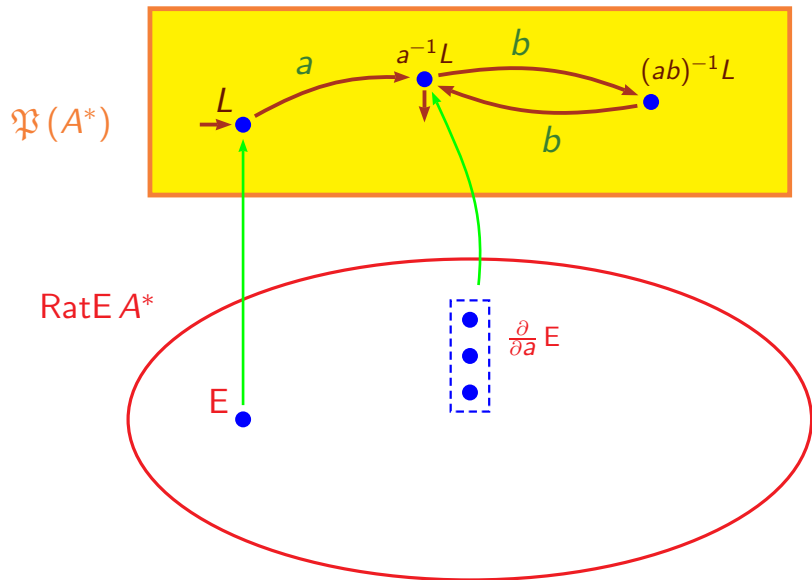
The Antimirov derived term automaton



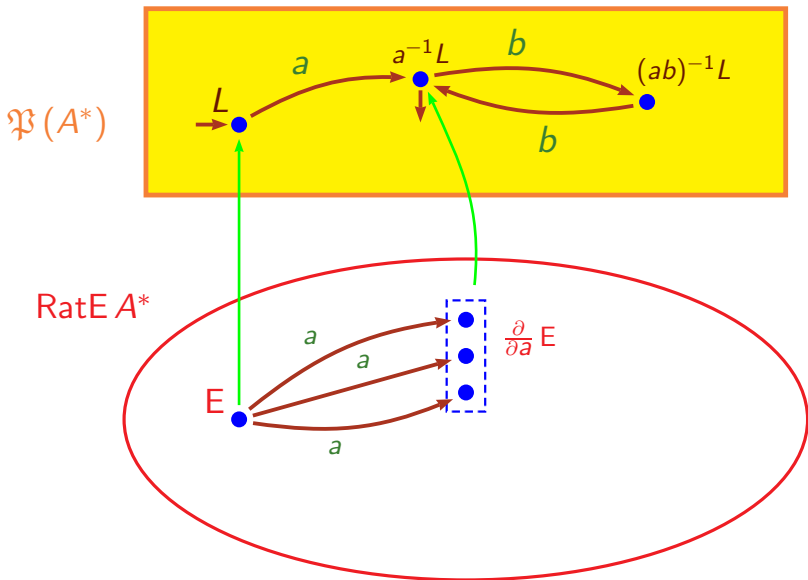
The Antimirov derived term automaton



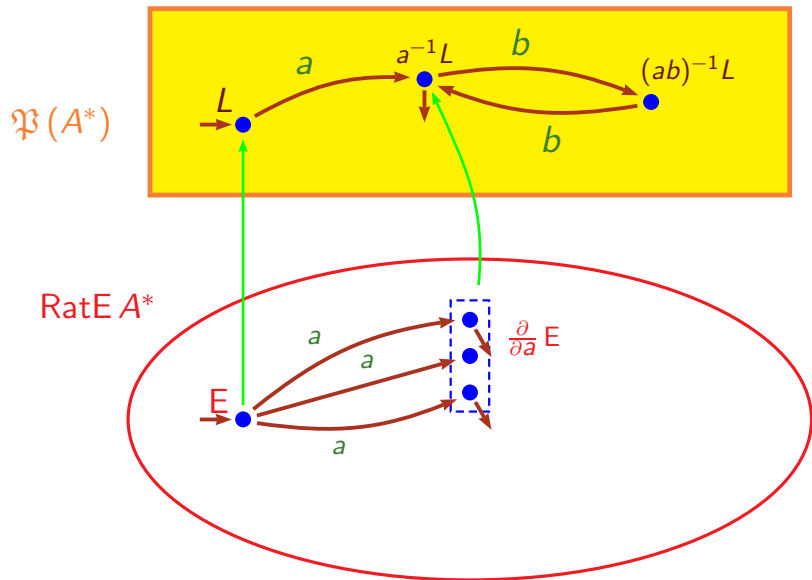
The Antimirov derived term automaton



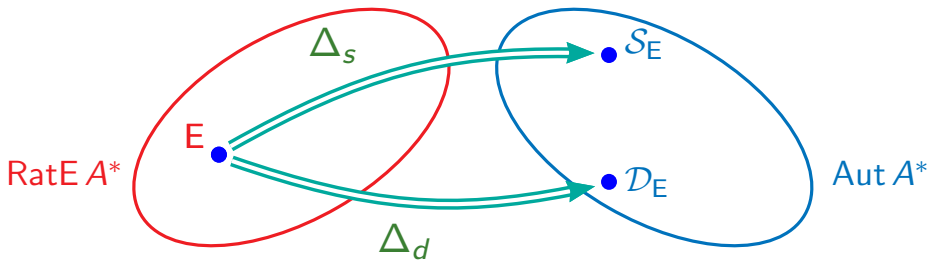
The Antimirov derived term automaton



The Antimirov derived term automaton



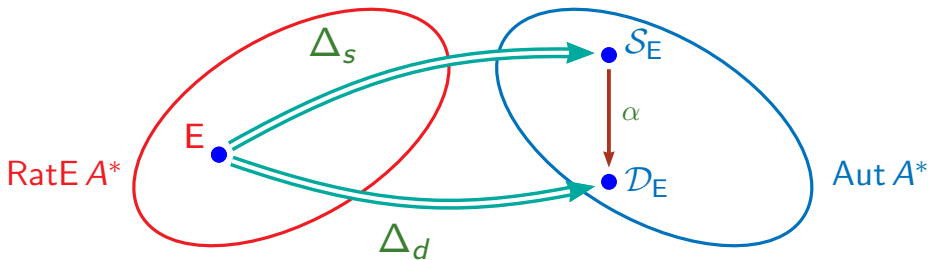
- ▶ Standard automaton of E position, Glushkov
- ▶ Derived term automaton of E Brzozowski–Antimirov



Th. [Glushkov 61] $|\mathcal{S}_E| = |E|$ $\dim \mathcal{S}_E = \ell(E) + 1$

Th. [Brzozowski–Antimirov 64–96] $|\mathcal{D}_E| = |E|$ $\dim \mathcal{D}_E \leq \ell(E) + 1$

- ▶ Standard automaton of E position, Glushkov
- ▶ Derived term automaton of E Brzozowski–Antimirov



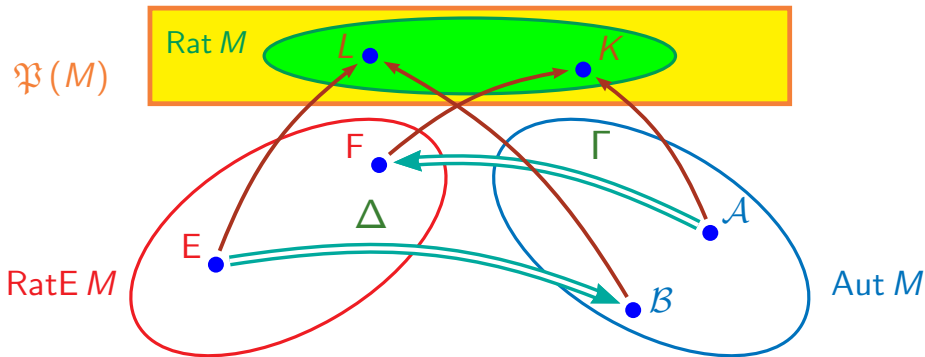
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Th. [Champarnaud-Ziadi 02] $\alpha: \mathcal{S}_E \rightarrow \mathcal{D}_E$ morphism of automata.

M non free monoid

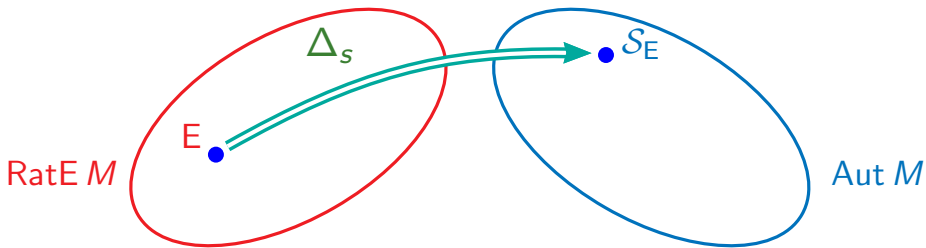
$\text{Rat } M \neq \text{Rec } M$



$$\forall n \in \mathbb{N} \quad (a^n, 1)^{-1}(a, b)^* = (1, b^n)(a, b)^*$$

► Standard automaton of E

position, Glushkov

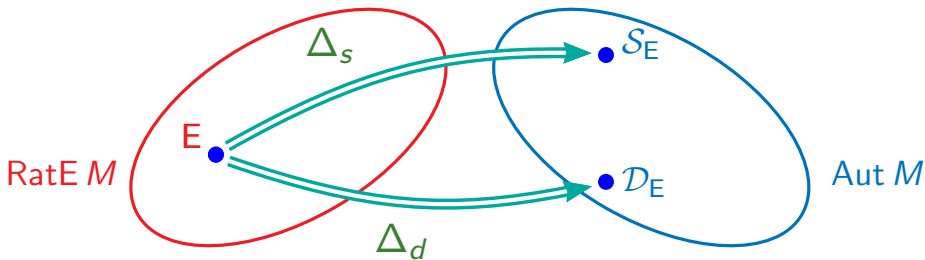


Th. [Glushkov 61]

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- ▶ Standard automaton of E position, Glushkov
- ▶ Derived term automaton of E Brzozowski–Antimirov

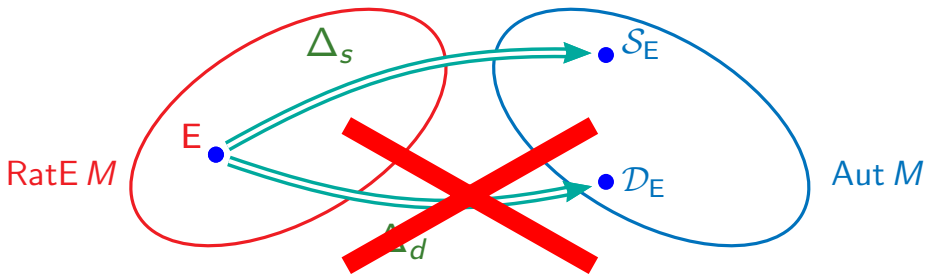


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► Standard automaton of E

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Th. [Glushkov 61]

$$|S_E| = |E|$$

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\mathbb{K} weight semiring

$\mathbb{K}\langle\langle A^* \rangle\rangle$



\mathbb{K} weight semiring eg $\mathbb{Z}, \mathbb{Q}, \mathbb{Z}_{\min}$

$\mathbb{K}\langle\langle A^* \rangle\rangle$



\mathbb{K} weight semiring

eg $\mathbb{Z}, \mathbb{Q}, \mathbb{Z}_{\min}$

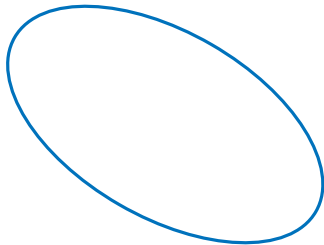
$\mathbb{K}\langle\langle A^* \rangle\rangle$



$\mathbb{K}\text{Rat}E A^*$

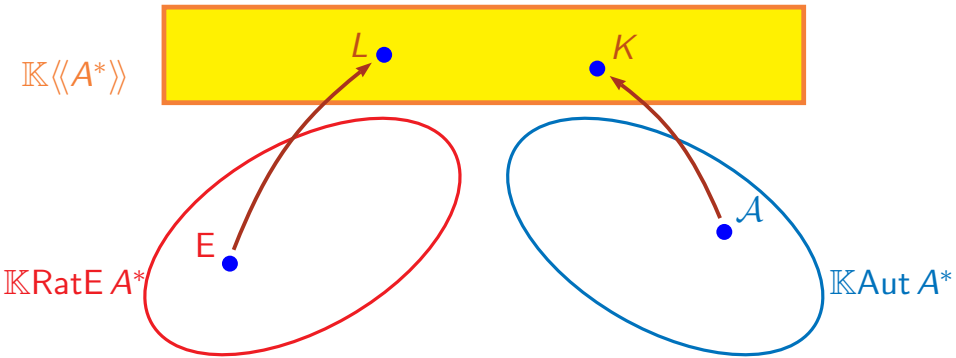


$\mathbb{K}\text{Aut} A^*$

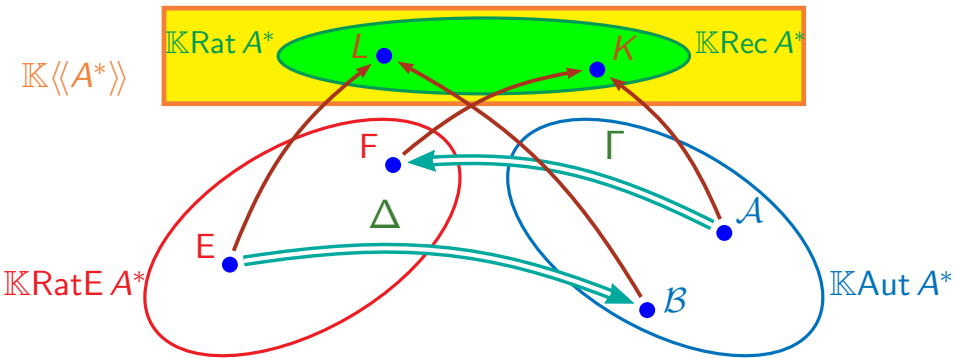


\mathbb{K} weight semiring

eg $\mathbb{Z}, \mathbb{Q}, \mathbb{Z}_{\min}$

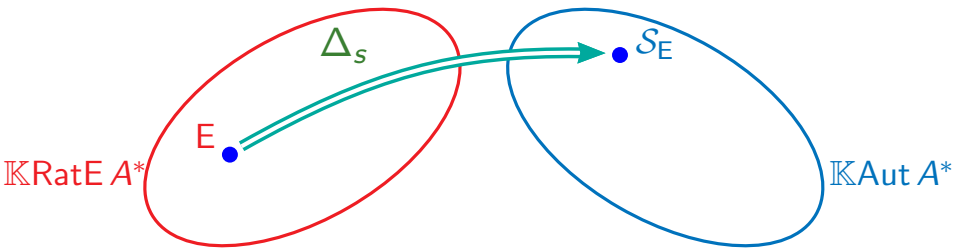


$$\mathbb{K}\text{Rat } A^* = \mathbb{K}\text{Rec } A^*$$



► Standard \mathbb{K} -automaton of E

position, Glushkov

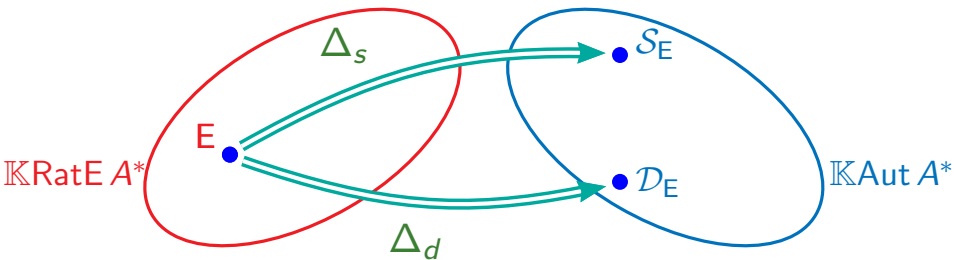


Th. [Caron-Flouret 00-03]

$$|S_E| = |E|$$

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- ▶ Standard \mathbb{K} -automaton of E position, Glushkov
- ▶ Derived term \mathbb{K} -automaton of E Brzozowski–Antimirov (?)



Th. [Caron-Flouret 00-03]

$$|\mathcal{S}_E| = |E|$$

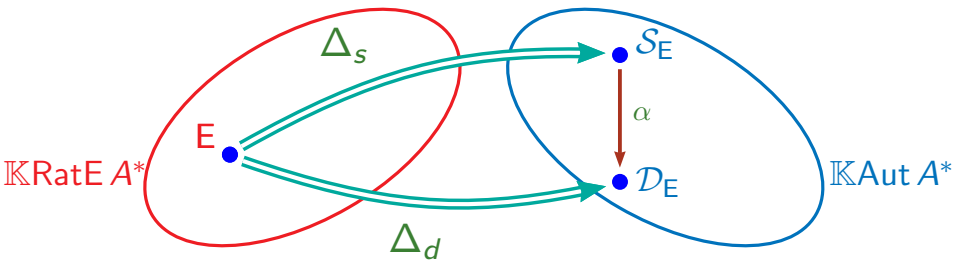
$$\dim \mathcal{S}_E = \ell(E) + 1$$

Th. [L.-S. 02-05]

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- ▶ Derived term \mathbb{K} -automaton of E Brzowski–Antimirov



- | | | |
|---------------------------|---|---------------------------------------|
| Th. [Caron-Flouret 00-03] | $ \mathcal{S}_E = E $ | $\dim \mathcal{S}_E = \ell(E) + 1$ |
| Th. [L.-S. 02-05] | $ \mathcal{D}_E = E $ | $\dim \mathcal{D}_E \leq \ell(E) + 1$ |
| Th. [L.-S. 03-05] | $\alpha: \mathcal{S}_E \rightarrow \mathcal{D}_E$ | morphism of \mathbb{K} -automata |

Inductive definition of derived terms

$$D(0) = D(1) = \emptyset \quad D(a) = \{1\}$$

$$D(F + G) = D(F) \cup D(G)$$

$$D(F \cdot G) = D(F) \cdot G \cup D(G)$$

$$D(F^*) = D(F) \cdot F^*$$

Inductive definition of derived terms

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By induction $\# D(E) \leq \ell(E)$

Inductive definition of the standard derived term automaton

$E \rightsquigarrow \mathcal{T}_E$ by induction on the formation of E

Inductive definition of the standard derived term automaton

$E \rightsquigarrow \mathcal{T}_E$ by induction on the formation of E

Property 1 $\dim \mathcal{T}_E = E \sqcup D(E)$

Property 2 $\beta: \mathcal{S}_E \rightarrow \mathcal{T}_E$ morphism

Inductive definition of the standard derived term automaton

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Base cases

$$\mathcal{T}_0 = \mathcal{S}_0$$

$$\mathcal{T}_1 = \mathcal{S}_1$$

$$\mathcal{T}_a = \mathcal{S}_a$$

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Addition

$$E = F + G$$

Inductive definition of the standard derived term automaton

$E \rightsquigarrow \mathcal{T}_E$ by induction on the formation of E

Property 1 $\dim \mathcal{T}_E = E \sqcup D(E)$

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Base cases

$$\mathcal{T}_0 = \mathcal{S}_0$$

$$\mathcal{T}_1 = \mathcal{S}_1$$

$$\mathcal{T}_a = \mathcal{S}_a$$

Addition

$$E = F + G$$

$$\dim(\mathcal{T}_F + \mathcal{T}_G) = E \sqcup D(F) \sqcup D(G)$$

$$\mathcal{A} + \mathcal{B} = \left\langle \left(1 \begin{array}{|c|c|} \hline 0 & 0 \\ \hline \end{array} \right), \left(\begin{array}{|c|c|c|} \hline 0 & J & K \\ \hline 0 & F & 0 \\ \hline 0 & 0 & G \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline c+d \\ \hline U \\ \hline V \\ \hline \end{array} \right) \right\rangle$$

Inductive definition of the standard derived term automaton

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$$\mathcal{A} \cdot \mathcal{B} = \left\langle \left(1 \begin{array}{|c|} \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline 0 \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline J \\ \hline \end{array} \begin{array}{|c|} \hline cK \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline F \\ \hline \end{array} \begin{array}{|c|} \hline U \cdot K \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline G \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline cd \\ \hline \end{array} \begin{array}{|c|} \hline Ud \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline V \\ \hline \end{array} \right) \right\rangle$$

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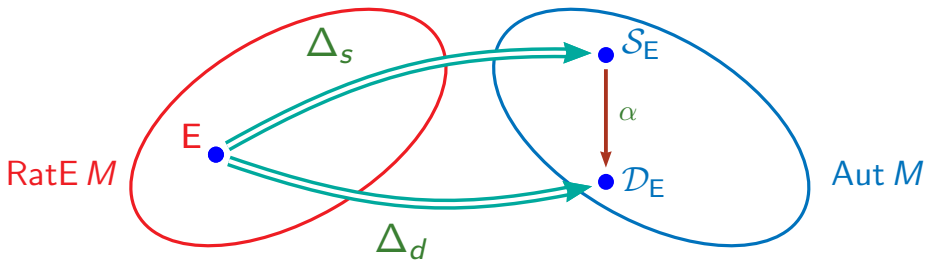
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Theorem

$\mathcal{D}_E = \omega(\mathcal{T}_E)$ is the derived term automaton of E

- ▶ Standard automaton of E position, Glushkov
- ▶ Derived term automaton of E Brzozowski–Antimirov



Th. [Glushkov 61] $|\mathcal{S}_E| = |E|$ $\dim \mathcal{S}_E = \ell(E) + 1$

Th. [L.-S.] $|\mathcal{D}_E| = |E|$ $\dim \mathcal{D}_E \leq \ell(E) + 1$

Th. [L.-S.] $\alpha: \mathcal{S}_E \rightarrow \mathcal{D}_E$ morphism of automata.