Derived terms without derivation

A shifted perspective on the derived-term automaton

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Joint work with

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Derived terms without derivation

Truth emerges from generalisation

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The results presented in this talk are based on a work that has been published recently in the *Journal of Computer Science and Cybernetics* 37 (2021) by the Vietnamese Academy of Science and Technology.

It can also be found at [https://arxiv.org/abs/2110.09181](https://arxiv.org/abs/2110.09181)
In the beginning, Kleene created the family of regular events and the family of recognizable events. He established that these two families coincide. He looked at it and thought it was good, and moved on to something else.
Rat $A^* = \text{Rec } A^*$
Rat $A^* = \text{Rec } A^*$

**Notations**

- $A$ *alphabet*, i.e. a finite set of *letters*
- $A^*$ set of *words*
- $L \subseteq A^*$ *language*
- Rat $A^*$ set of *regular languages* over $A^*$
  - RatE $A^*$ set of *regular expressions* over $A^*$
  - $|E|$ *language denoted* by the expression $E$
- Rec $A^*$ set of *recognizable languages* over $A^*$
- Aut $A^*$ set of *finite automata* over $A^*$
  - $|A|$ *language accepted* by the automaton $A$
Rat $A^* = \text{Rec } A^*$
Rat $A^* = \text{Rec } A^*$
$M$ non free monoid \hspace{1cm} \text{Rat} \ M \neq \text{Rec} \ M$

\[\text{Rat} \ M \ \\
\mathcal{P}(M) \ \\
\text{RatE} \ M \ \\
\text{Aut} \ M\]
$\mathcal{K} \text{Rat} A^* = \mathcal{K} \text{Rec} A^*$
$M$ non free monoid \quad $\mathbb{K}\text{Rat} M \neq \mathbb{K}\text{Rec} M$
\[ \mathcal{P}(A^*) \]

\[ \text{Rat } A^* = \text{Rec } A^* \]
- Standard automaton of $E$ position, Glushkov

Th. [Glushkov 61]

$|S_E| = |E|$

$\dim S_E = \ell(E) + 1$
- Standard automaton of $E$ position, Glushkov
- Derived term automaton of $E$ Brzozowski–Antimirov

\[ \text{RatE } A^* \]

\[ \Delta_s \]

\[ \Delta_d \]

\[ S_E \]

\[ D_E \]

\[ \text{Aut } A^* \]

Th. [Glushkov 61] \[ |S_E| = |E| \quad \text{dim } S_E = \ell(E) + 1 \]

Th. [Brzozowski–Antimirov 64–96] \[ |D_E| = |E| \quad \text{dim } D_E \leq \ell(E) + 1 \]
- Standard automaton of $E$ position, Glushkov
- Derived term automaton of $E$ Brzozowski–Antimirov

Th. [Glushkov 61] $|S_E| = |E|$ \quad $\dim S_E = \ell(E) + 1$

Th. [Brzozowski–Antimirov 64–96] $|D_E| = |E|$ \quad $\dim D_E \leq \ell(E) + 1$

Th. [Champarnaud-Ziadi 02] $\alpha: S_E \to D_E$ morphism of automata.
- Standard automaton of $E$ (position, Glushkov)
- Derived term automaton of $E$ (Brzozowski–Antimirov)

Th. [Glushkov 61] \[ |S_E| = |E| \quad \dim S_E = \ell(E) + 1 \]

Th. [Brzozowski–Antimirov 64–96] \[ |D_E| = |E| \quad \dim D_E \leq \ell(E) + 1 \]

Th. [Champarnaud-Ziadi 02] $\alpha: S_E \to D_E$ morphism of automata.
The standard automaton of an expression

Definition of an automaton

\[ A = \langle I, E, T \rangle \]
The standard automaton of an expression

Definition of an automaton

\[ \mathcal{A} = \left\langle \left( \begin{array} { l } { I } \end{array} \right), \left( \begin{array} { l } { E } \end{array} \right), \left( \begin{array} { l } { T } \end{array} \right) \right\rangle \]
The standard automaton of an expression

Definition of an automaton

\[ \mathcal{A} = \langle (I), (E), (T) \rangle \]

\[ \mathcal{A} = \langle (1 1 0), \begin{pmatrix} a & b & b \\ a & 0 & a+b \\ a & 0 & b \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \rangle \]
The standard automaton of an expression

Definition of an automaton

\[ A = \left\langle (\square_1), (\square_E), (\square_T) \right\rangle \]

The language accepted by \( A \)

\[ |A| = I \cdot E^* \cdot T \]
The standard automaton of an expression

Definition of a standard automaton

\[ \mathcal{A} = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & J \\ 0 & F \end{pmatrix}, \begin{pmatrix} c \\ U \end{pmatrix} \right\rangle \]
The standard automaton of an expression

Definition of a standard automaton

\[ A = \langle (1, 0), \begin{pmatrix} 0 & J \\ 0 & F \end{pmatrix}, \begin{pmatrix} c \\ U \end{pmatrix} \rangle \]

The language accepted by \( A \)

\[ |A| = c + J \cdot F^* \cdot U \]
The standard automaton of an expression

Definition of a standard automaton

\[ A = \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & J \\ 0 & F \end{pmatrix}, \begin{pmatrix} c \\ U \end{pmatrix} \rangle \]

Operations on standard automata

\[ A + B \quad A \cdot B \quad A^* \]
The standard automaton of an expression

Definition of a standard automaton

\[ \mathcal{A} = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & J \end{pmatrix}, \begin{pmatrix} 0 & F \\ 0 & \end{pmatrix}, \begin{pmatrix} c \\ U \end{pmatrix} \right\rangle \]

Operations on standard automata

\[ \mathcal{A} + \mathcal{B} \quad \mathcal{A} \cdot \mathcal{B} \quad \mathcal{A}^* \]
The standard automaton of an expression

Definition of a standard automaton

\[ A = \left\langle (1 \begin{pmatrix} 0 \\ \hline 0 & F \end{pmatrix}), \begin{pmatrix} c \\ U \end{pmatrix} \right\rangle \]

Operations on standard automata

\[ A + B \]
\[ |A + B| = |A| \cup |B| \]
\[ A \cdot B \]
\[ A^* \]
The standard automaton of an expression

Definition of a standard automaton

\[ A = \langle (1 \ 0) , \begin{pmatrix} 0 & J \\ 0 & F \end{pmatrix} , \begin{pmatrix} c \\ U \end{pmatrix} \rangle \]

Operations on standard automata

\[ A + B \]
\[ A + B = A \cup B \]
\[ |A + B| = |A| \cup |B| \]

\[ A \cdot B \]

\[ A^* \]

\[ A + B = \langle (1 \ \begin{pmatrix} 0 & 0 \end{pmatrix}) , \begin{pmatrix} 0 & 0 \\ 0 & F \\ 0 & 0 \end{pmatrix} , \begin{pmatrix} c + d \\ U \\ V \end{pmatrix} \rangle \]
The standard automaton of an expression

Definition of a standard automaton

\[ \mathcal{A} = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & F \end{pmatrix}, \begin{pmatrix} 0 & J \\ 0 & F \end{pmatrix}, \begin{pmatrix} c \\ U \end{pmatrix} \right\rangle \]

Operations on standard automata

\[ \mathcal{A} + \mathcal{B} \]

\[ |\mathcal{A} + \mathcal{B}| = |\mathcal{A}| \cup |\mathcal{B}| \]
The standard automaton of an expression

Definition of a standard automaton

\[ A = \langle (1, 0), \begin{pmatrix} 0 & J \\ 0 & F \end{pmatrix}, \begin{pmatrix} c \\ U \end{pmatrix} \rangle \]

Operations on standard automata

\[ A + B \]
\[ |A + B| = |A| \cup |B| \]

\[ A \cdot B \]
\[ |A \cdot B| = |A| \cdot |B| \]

\[ A^* \]
The standard automaton of an expression

Definition of a standard automaton

\[ A = \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & J \\ 0 & F \end{pmatrix}, \begin{pmatrix} c \\ U \end{pmatrix} \rangle \]

Operations on standard automata

\[ A + B = A \cup B \]  \[ A \cdot B = A \cdot B \]  \[ A^* \]

\[ |A + B| = |A| \cup |B| \]  \[ |A \cdot B| = |A| \cdot |B| \]  

\[ A \cdot B = \langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & F & U \cdot K \\ 0 & 0 & G \end{pmatrix}, \begin{pmatrix} c & d \\ Ud & V \end{pmatrix} \rangle \]
The standard automaton of an expression

Definition of a standard automaton

\[ A = \left( \begin{pmatrix} 1 & 0 \\ 0 & J \end{pmatrix}, \begin{pmatrix} 0 & F \\ 0 & U \end{pmatrix}, \begin{pmatrix} c \end{pmatrix} \right) \]

Operations on standard automata

\[ A + B, \quad A \cdot B, \quad A^* \]

\[ |A + B| = |A| \cup |B|, \quad |A \cdot B| = |A| \cdot |B| \]
The standard automaton of an expression

Definition of a standard automaton

\[ \mathcal{A} = \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & J \\ 0 & F \end{pmatrix}, \begin{pmatrix} c \\ U \end{pmatrix} \right) \]

Operations on standard automata

\[ \mathcal{A} + \mathcal{B} \]
\[ |\mathcal{A} + \mathcal{B}| = |\mathcal{A}| \cup |\mathcal{B}| \]

\[ \mathcal{A} \cdot \mathcal{B} \]
\[ |\mathcal{A} \cdot \mathcal{B}| = |\mathcal{A}| \cdot |\mathcal{B}| \]

\[ \mathcal{A}^* \]
\[ |\mathcal{A}^*| = |\mathcal{A}|^* \]
The standard automaton of an expression

Definition of a standard automaton

\[ \mathcal{A} = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & J \end{pmatrix}, \begin{pmatrix} 0 & F \\ 0 & H \end{pmatrix}, \begin{pmatrix} c \\ U \end{pmatrix} \right\rangle \]

Operations on standard automata

\[ \mathcal{A} + \mathcal{B} \quad \mathcal{A} \cdot \mathcal{B} \quad \mathcal{A}^* \]

\[ |\mathcal{A} + \mathcal{B}| = |\mathcal{A}| \cup |\mathcal{B}| \quad |\mathcal{A} \cdot \mathcal{B}| = |\mathcal{A}| \cdot |\mathcal{B}| \quad |\mathcal{A}^*| = |\mathcal{A}|^* \]

\[ \mathcal{A}^* = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & J \end{pmatrix}, \begin{pmatrix} 0 & F \\ 0 & H \end{pmatrix}, \begin{pmatrix} 1 \\ U \end{pmatrix} \right\rangle \]

with \( H = U \cdot J + F \)
The standard automaton of an expression

Definition of a standard automaton

\[ \mathcal{A} = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & F \end{pmatrix}, \begin{pmatrix} 0 & J \\ 0 & F \end{pmatrix}, \begin{pmatrix} c \\ U \end{pmatrix} \right\rangle \]

Operations on standard automata

\[ \mathcal{A} + \mathcal{B} \quad \mathcal{A} \cdot \mathcal{B} \quad \mathcal{A}^* \]

\[ |\mathcal{A} + \mathcal{B}| = |\mathcal{A}| \cup |\mathcal{B}| \quad |\mathcal{A} \cdot \mathcal{B}| = |\mathcal{A}| \cdot |\mathcal{B}| \quad |\mathcal{A}^*| = |\mathcal{A}|^* \]

Definition of \( \mathcal{S}_E \)
The standard automaton of an expression

Definition of a standard automaton

\[ A = \langle \begin{pmatrix} 1 & 0 \\ 0 & F \end{pmatrix}, \begin{pmatrix} 0 & J \\ 0 & F \end{pmatrix}, \begin{pmatrix} c \\ U \end{pmatrix} \rangle \]

Definition of \( S_E \)

Start from

\( S_0 = i \), \( S_1 = i \), and \( S_a = i \)
The standard automaton of an expression

Definition of a standard automaton

\[ \mathcal{A} = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & F \end{pmatrix}, \begin{pmatrix} 0 & J \\ 0 & F \end{pmatrix}, \begin{pmatrix} c \\ U \end{pmatrix} \right\rangle \]

Definition of \( S_E \)

Start from

\[ S_0 = \left\langle (1), (0), (0) \right\rangle, \ S_1 = \left\langle (1), (0), (1) \right\rangle, \ \text{and} \ S_a = \left\langle (1, 0), \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \]
The standard automaton of an expression

Definition of a standard automaton

\[ \mathcal{A} = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & F \end{pmatrix}, \begin{pmatrix} 0 & J \\ 0 & F \end{pmatrix}, \begin{pmatrix} c \\ U \end{pmatrix} \right\rangle \]

Definition of \( S_E \)

Start from

\[ S_0 = \left\langle (1), (0), (0) \right\rangle, \quad S_1 = \left\langle (1), (0), (1) \right\rangle, \quad \text{and} \quad S_a = \left\langle (1, 0), \begin{pmatrix} 0 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \]

Define \( S_{F+G} = S_F + S_G \), \( S_{F \cdot G} = S_F \cdot S_G \), and \( S_F^* = (S_F)^* \)
The standard automaton of an expression

Definition of a standard automaton

\[ \mathcal{A} = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & J \end{pmatrix}, \begin{pmatrix} 0 & F \\ 0 & \mathcal{U} \end{pmatrix}, \begin{pmatrix} c \end{pmatrix} \right\rangle \]

Definition of \( S_E \)

Start from

\[ S_0 = \left\langle (1), (0), (0) \right\rangle, \quad S_1 = \left\langle (1), (0), (1) \right\rangle, \quad \text{and} \quad S_a = \left\langle (1, 0), \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \]

Define \( S_{F+G} = S_F + S_G \), \( S_{F \cdot G} = S_F \cdot S_G \), and \( S_F^* = (S_F)^* \)

And get \( S_E = \Delta_s(E) \)
Standard automaton of $E$ position, Glushkov

Th. [Glushkov 61]

$|S_E| = |E| \quad \text{dim} \ S_E = \ell(E) + 1$
Preliminary

▶ Th. A language $L$ is *recognisable* iff it has a *finite* number of (left) *quotients*.

▶ Construction of $A_L$, the *minimal (deterministic) automaton* of $L$ by means of the *quotients* of $L$

$$u^{-1}L = \{v \in A^* \mid uv \in L\}$$
The Brzozowski derivative automaton

\[ \mathfrak{B}(A^*) \]
The Brzozowski derivative automaton

\[ \mathcal{P}(A^*) \]

\[ L \quad \overset{a}{\longrightarrow} \quad \overset{a^{-1}L}{\longrightarrow} \quad \overset{b}{\longrightarrow} \quad \overset{(ab)^{-1}L}{\longrightarrow} \quad \]
The Brzozowski derivative automaton

\[ P(A^*) \]

\[ L \]

\[ a \]

\[ a^{-1}L \]

\[ b \]

\[ (ab)^{-1}L \]

\[ \Psi(A^*) \]

\[ \text{RatE } A^* \]

\[ E \]
The Brzozowski derivative automaton

\[ P(A^*) \]

\[ L \bullet \quad a \quad (ab)^{-1}L \quad b \]

\[ \mathcal{P}(A^*) \]

\[ \text{RatE } A^* \]

\[ \frac{\partial}{\partial a} E \]
The Brzozowski derivative automaton

\[ P(A^*) / Bullet \ L \ a^{-1}L \ / Bullet \ (ab)^{-1}L \]

\[ \mathcal{V} (A^*) \]

\[ \text{RatE } A^* \]

\[ \frac{\partial}{\partial a} E \]
The Brzozowski derivative automaton

\[ \mathcal{A}(A^*) \]

\[ L \]

\[ a \]

\[ a^{-1}L \]

\[ b \]

\[ (ab)^{-1}L \]

\[ \frac{\partial}{\partial a} E \]

\[ a \]

\[ E \]
The Brzozowski derivative automaton

$\mathfrak{B}(A^*)$

$P(A^*)/\bullet L/a^{-1}L/(ab)^{-1}L$

$\Psi(A^*)/\bullet E/\bullet E/\partial a/\partial ab$

RatE $A^*$
The Brzozowski derivative automaton

\[ \mathcal{B}(A^*) \]

\[ L \]

\[ a \]

\[ a^{-1}L \]

\[ b \]

\[ b \]

\[ (ab)^{-1}L \]

RatE A*

\[ E \]

\[ \frac{\partial}{\partial a} E \]

\[ \frac{\partial}{\partial ab} E \]

\[ \frac{\partial}{\partial ab} E \]

\[ \frac{\partial}{\partial a} E \]
The Brzozowski derivative automaton

\[ \mathfrak{P}(A^*) \]

\[ \mathfrak{Y}(A^*) \]

\[ \text{RatE } A^* \]

\[ \frac{\partial}{\partial a} E \]

\[ \frac{\partial}{\partial ab} E \]

\[ \frac{\partial}{\partial abb} E \]
The Brzozowski derivative automaton

\[ P(A^*) ]

\[ L \]

\[ a^{-1}L \]

\[ b \]

\[ (ab)^{-1}L \]

\[ \mathcal{P}(A^*) \]

RatE \( A^* \)

\[ \frac{\partial}{\partial a} E \]

\[ a \]

\[ \frac{\partial}{\partial ab} E \]

\[ b \]

\[ \frac{\partial}{\partial ab} E \]

\[ b \]

\[ \frac{\partial}{\partial abbb} E \]
The Brzozowski derivation

Definition (Brzozowski 64)

\[ E \in \text{Rat} \exists A^* \quad \frac{\partial}{\partial a} E \text{ by induction on } E \]

\[
\frac{\partial}{\partial a} 0 = \frac{\partial}{\partial a} 1 = 0, \quad \frac{\partial}{\partial a} b = \begin{cases} 
1 & \text{if } b = a \\
0 & \text{otherwise}
\end{cases}
\]

\[
\frac{\partial}{\partial a} (F + G) = \frac{\partial}{\partial a} F + \frac{\partial}{\partial a} G
\]

\[
\frac{\partial}{\partial a} (F \cdot G) = \left[ \frac{\partial}{\partial a} F \right] \cdot G + c(F) \frac{\partial}{\partial a} G
\]

\[
\frac{\partial}{\partial a} (F^*) = \left[ \frac{\partial}{\partial a} F \right] \cdot F^*
\]
The Brzozowski derivation

Definition (Brzozowski 64)

\[ E \in \text{RatE} \quad A^* \quad \frac{\partial}{\partial a} E \quad \text{by induction on } E \quad \frac{\partial}{\partial u} E = \frac{\partial}{\partial a} \left( \frac{\partial}{\partial u} E \right) \]

\[ \frac{\partial}{\partial a} 0 = \frac{\partial}{\partial a} 1 = 0, \quad \frac{\partial}{\partial a} b = \begin{cases} 1 & \text{if } b = a \\ 0 & \text{otherwise} \end{cases} \]

\[ \frac{\partial}{\partial a} (F + G) = \frac{\partial}{\partial a} F + \frac{\partial}{\partial a} G \]

\[ \frac{\partial}{\partial a} (F \cdot G) = \left[ \frac{\partial}{\partial a} F \right] \cdot G + c(F) \frac{\partial}{\partial a} G \]

\[ \frac{\partial}{\partial a} (F^*) = \left[ \frac{\partial}{\partial a} F \right] \cdot F^* \]

Theorem (Brzozowski 64)

For every \( E \), there is a finite number of derivatives modulo \( A \), \( C \), and \( I \).
- Standard automaton of $E$ position, Glushkov
- Derivative automaton of $E$ Brzozowski

\[ \text{Th. [Glushkov 61]} \quad |S_E| = |E| \quad \dim S_E = \ell(E) + 1 \]

\[ \text{Th. [Brzozowski 64]} \quad |\mathcal{B}_E| = |E|. \]
The Brzozowski–Antimirov derivation

Definition (Brzozowski 64)

\( E \in \text{RatE} \) \( A^* \) \( \frac{\partial}{\partial a} E \) by induction on \( E \)

\[
\frac{\partial}{\partial a} 0 = \frac{\partial}{\partial a} 1 = 0, \quad \frac{\partial}{\partial a} b = \begin{cases} 1 & \text{if } b = a \\ 0 & \text{otherwise} \end{cases}
\]

\[
\frac{\partial}{\partial a} (F + G) = \frac{\partial}{\partial a} F + \frac{\partial}{\partial a} G
\]

\[
\frac{\partial}{\partial a} (F \cdot G) = \left[ \frac{\partial}{\partial a} F \right] \cdot G + c(F) \frac{\partial}{\partial a} G
\]

\[
\frac{\partial}{\partial a} (F^*) = \left[ \frac{\partial}{\partial a} F \right] \cdot F^*
\]
The Brzozowski–Antimirov derivation

Definition (Brzozowski 64 — Antimirov 96)

\( E \in \text{RatE} \quad A^* \quad \frac{\partial}{\partial a} \quad E \) by induction on \( E \)

\[
\frac{\partial}{\partial a} 0 = \frac{\partial}{\partial a} 1 = 0, \quad \frac{\partial}{\partial a} b = \begin{cases} 1 & \text{if } b = a \\ 0 & \text{otherwise} \end{cases}
\]

\[
\frac{\partial}{\partial a} (F + G) = \frac{\partial}{\partial a} F \cup \frac{\partial}{\partial a} G
\]

\[
\frac{\partial}{\partial a} (F \cdot G) = \left[ \frac{\partial}{\partial a} F \right] \cdot G \cup c(F) \frac{\partial}{\partial a} G
\]

\[
\frac{\partial}{\partial a} (F^*) = \left[ \frac{\partial}{\partial a} F \right] \cdot F^*
\]
The Brzozowski–Antimirov derivation

Definition (Brzozowski 64 — Antimirov 96)

\( E \in \text{Rat}E A^* \) \( \frac{\partial}{\partial a} E \) by induction on \( E \)

\[
\frac{\partial}{\partial a} 0 = \frac{\partial}{\partial a} 1 = 0, \quad \frac{\partial}{\partial a} b = \begin{cases} 
1 & \text{if } b = a \\
0 & \text{otherwise}
\end{cases}
\]

\[
\frac{\partial}{\partial a} (F + G) = \frac{\partial}{\partial a} F \cup \frac{\partial}{\partial a} G
\]

\[
\frac{\partial}{\partial a} (F \cdot G) = \left[ \frac{\partial}{\partial a} F \right] \cdot G \cup c(F) \frac{\partial}{\partial a} G
\]

\[
\frac{\partial}{\partial a} (F^*) = \left[ \frac{\partial}{\partial a} F \right] \cdot F^*
\]

\[
\frac{\partial}{\partial a} \left[ \bigcup_{i \in I} F_i \right] = \bigcup_{i \in I} \frac{\partial}{\partial a} F_i, \quad \left[ \bigcup_{i \in I} F_i \right] \cdot G = \bigcup_{i \in I} (F_i \cdot G).
\]
The Brzozowski–Antimirov derivation

Definition (Brzozowski 64 — Antimirov 96)

\( \mathbb{E} \in \text{Rat} \mathbb{E} \mathbb{A}^* \) \( \frac{\partial}{\partial a} \mathbb{E} \) by induction on \( \mathbb{E} \)

\[
\frac{\partial}{\partial a} 0 = \frac{\partial}{\partial a} 1 = 0 , \quad \frac{\partial}{\partial a} b = \begin{cases} 
1 & \text{if } b = a \\
0 & \text{otherwise}
\end{cases}
\]

\[
\frac{\partial}{\partial a} (\mathbb{F} + \mathbb{G}) = \frac{\partial}{\partial a} \mathbb{F} \cup \frac{\partial}{\partial a} \mathbb{G}
\]

\[
\frac{\partial}{\partial a} (\mathbb{F} \cdot \mathbb{G}) = \left[ \frac{\partial}{\partial a} \mathbb{F} \right] \cdot \mathbb{G} \cup \text{c} (\mathbb{F}) \frac{\partial}{\partial a} \mathbb{G}
\]

\[
\frac{\partial}{\partial a} (\mathbb{F}^*) = \left[ \frac{\partial}{\partial a} \mathbb{F} \right] \cdot \mathbb{F}^*
\]

Theorem (Antimirov 96)

For every \( \mathbb{E} \), there is a finite number of derived terms
The Brzozowski–Antimirov derivation

Definition (Brzozowski 64 — Antimirov 96)

\[ E \in \text{Rat}E A^* \quad \frac{\partial}{\partial a} E \text{ by induction on } E \quad \frac{\partial}{\partial u a} E = \frac{\partial}{\partial a} \left( \frac{\partial}{\partial u} E \right) \]

\[ \frac{\partial}{\partial a} 0 = \frac{\partial}{\partial a} 1 = 0 \]
\[ \frac{\partial}{\partial a} b = \begin{cases} 1 & \text{if } b = a \\ 0 & \text{otherwise} \end{cases} \]

\[ \frac{\partial}{\partial a} (F + G) = \frac{\partial}{\partial a} F \cup \frac{\partial}{\partial a} G \]

\[ \frac{\partial}{\partial a} (F \cdot G) = \left[ \frac{\partial}{\partial a} F \right] \cdot G \cup c(F) \frac{\partial}{\partial a} G \]

\[ \frac{\partial}{\partial a} (F^*) = \left[ \frac{\partial}{\partial a} F \right] \cdot F^* \]

Theorem (Antimirov 96)

For every \( E \), there is a finite number of derived terms

\( (\leq \ell(E) + 1) \)
The Antimirov derived term automaton

\[ \mathfrak{P}(A^*) \]

\[ \text{RatE } A^* \]
The Antimirov derived term automaton

\[ \mathfrak{P}(A^*) \]

\[ \mathfrak{R} \text{atE } A^* \]

\[ \text{E} \]

\[ \frac{\partial}{\partial a} \text{ E} \]
The Antimirov derived term automaton

$\mathfrak{P}(A^*)$
The Antimirov derived term automaton

\[ \mathcal{P}(A^*) \]

\[ \text{RatE } A^* \]

\[ \frac{\partial}{\partial a} E \]
The Antimirov derived term automaton

\[ \Psi(A^*) \]

\[ \text{RatE } A^* \]
The Antimirov derived term automaton

\[ \mathfrak{P}(A^*) \]

\[ \mathfrak{RatE} A^* \]
Standard automaton of $E$ position, Glushkov

Derived term automaton of $E$ Brzozowski–Antimirov

$|S_E| = |E| \quad \dim S_E = \ell(E) + 1$

$|D_E| = |E| \quad \dim D_E \leq \ell(E) + 1$
- Standard automaton of $E$ position, Glushkov
- Derived term automaton of $E$ Brzozowski–Antimirov

\[ \begin{align*}
|S_E| &= |E| \\
\dim S_E &= \ell(E) + 1 \\
|D_E| &= |E| \\
\dim D_E &\leq \ell(E) + 1 \\
\alpha : S_E &\to D_E \text{ morphism of automata.}
\end{align*} \]
\( \forall n \in \mathbb{N} \quad (a^n, 1)^{-1}(a, b)^* = (1, b^n)(a, b)^* \)
Standard automaton of $E$ position, Glushkov

Th. [Glushkov 61] $|S_E| = |E|$ \hspace{1cm} \dim S_E = \ell(E) + 1
- Standard automaton of $E$ position, Glushkov
- Derived term automaton of $E$ Brzozowski–Antimirov

\[ |S_E| = |E| \]
\[ \dim S_E = \ell(E) + 1 \]

\[ |D_E| = |E| \]
\[ \dim D_E \leq \ell(E) + 1 \]
Standard automaton of \(E\) position, Glushkov

\[ S_E = |E| \]
\[ \dim S_E = \ell(E) + 1 \]
$K$-weight semiring

$K \langle \langle A^* \rangle \rangle$
$K$ weight semiring eg $\mathbb{Z}, \mathbb{Q}, \mathbb{Z}_{\min}$
$K$ weight semiring eg $\mathbb{Z}, \mathbb{Q}, \mathbb{Z}_{\text{min}}$

$K \langle A^* \rangle$

$KRatE A^*$

$KAut A^*$
$K$ weight semiring eg $\mathbb{Z}, \mathbb{Q}, \mathbb{Z}_{\text{min}}$

$K\langle A^* \rangle$

$KR\text{at}E A^*$

$K\text{Aut} A^*$
$$\mathbb{KRat} A^* = \mathbb{KRec} A^*$$
Standard $K$-automaton of $E$ position, Glushkov

\[ S_E = E \text{ dim } S_E = \ell(E) + 1 \]

Th. [Caron-Flouret 00-03] \[ |S_E| = |E| \quad \text{dim } S_E = \ell(E) + 1 \]
- Standard $K$-automaton of $E$ position, Glushkov
- Derived term $K$-automaton of $E$ Brzozowski–Antimirov (?)

Th. [Caron-Flouret 00-03] $|S_E| = |E|$  \hspace{1cm} \text{dim } S_E = \ell(E) + 1$
Th. [L.–S. 02-05] $|D_E| = |E|$  \hspace{1cm} \text{dim } D_E \leq \ell(E) + 1$
- Standard $K$-automaton of $E$ position, Glushkov
- Derived term $K$-automaton of $E$ Brzozowski–Antimirov

\[ S_E = |E| \quad \text{dim} \, S_E = \ell(E) + 1 \]

\[ D_E = |E| \quad \text{dim} \, D_E \leq \ell(E) + 1 \]

\[ \alpha : S_E \to D_E \quad \text{morphism of } K\text{-automata} \]
Inductive definition of derived terms

\[ D(0) = D(1) = \emptyset \quad D(a) = \{1\} \]

\[ D(F + G) = D(F) \cup D(G) \]

\[ D(F \cdot G) = D(F) \cdot G \cup D(G) \]

\[ D(F^*) = D(F) \cdot F^* \]
Inductive definition of derived terms

\[
\begin{align*}
D(0) &= D(1) = \emptyset \\
D(a) &= \{1\} \\
D(F + G) &= D(F) \cup D(G) \\
D(F \cdot G) &= D(F) \cdot G \cup D(G) \\
D(F^*) &= D(F) \cdot F^*
\end{align*}
\]

By induction \# D(E) \leq \ell(E)
Inductive definition of the standard derived term automaton

\[ E \rightsquigarrow \mathcal{T}_E \]  by induction on the formation of  \( E \)
Inductive definition of the standard derived term automaton

\[ E \leadsto \mathcal{T}_E \] by induction on the formation of \( E \)

Property 1  \( \dim \mathcal{T}_E = E \sqcup D(E) \)

Property 2  \( \beta : S_E \rightarrow \mathcal{T}_E \) morphism
Inductive definition of the standard derived term automaton

\[ E \leadsto \mathcal{T}_E \] by induction on the formation of \( E \)

Property 1 \( \dim \mathcal{T}_E = E \sqcup D(E) \)

Property 2 \( \beta : S_E \rightarrow \mathcal{T}_E \) morphism

Base cases
\[ \mathcal{T}_0 = S_0 \quad \mathcal{T}_1 = S_1 \quad \mathcal{T}_a = S_a \]
Inductive definition of the standard derived term automaton

\[ E \leadsto \mathcal{T}_E \] by induction on the formation of \( E \)

Property 1 \( \dim \mathcal{T}_E = E \sqcup D(E) \)  
Property 2 \( \beta : S_E \to \mathcal{T}_E \) morphism

Base cases  
\[ \mathcal{T}_0 = S_0 \quad \mathcal{T}_1 = S_1 \quad \mathcal{T}_a = S_a \]

Addition  
\[ E = F + G \]
Inductive definition of the standard derived term automaton

$E \leadsto \mathcal{T}_E$ by induction on the formation of $E$

Property 1 $\dim \mathcal{T}_E = E \sqcup D(E)$

Property 2 $\beta : S_E \rightarrow \mathcal{T}_E$ morphism

Base cases

$\mathcal{T}_0 = S_0$

$\mathcal{T}_1 = S_1$

$\mathcal{T}_a = S_a$

Addition $E = F + G$

$\dim(\mathcal{T}_F + \mathcal{T}_G) = E \sqcup D(F) \sqcup D(G)$

$\mathcal{A} + \mathcal{B} = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & J & K \\ 0 & F & 0 \\ 0 & 0 & G \\ 0 & U & V \end{pmatrix} \right\rangle$
Inductive definition of the standard derived term automaton

\[ E \leadsto \mathcal{T}_E \] by induction on the formation of \( E \)

\[
\begin{align*}
D(F + G) &= D(F) \cup D(G) \\
D(F \cdot G) &= D(F) \cdot G \cup D(G) \\
D(F^*) &= D(F) \cdot F^*
\end{align*}
\]

Property 1 \( \dim \mathcal{T}_E = E \sqcup D(E) \) \hspace{1cm} Property 2 \( \beta : S_E \to \mathcal{T}_E \) morphism

Base cases \( \mathcal{T}_0 = S_0 \), \( \mathcal{T}_1 = S_1 \), \( \mathcal{T}_a = S_a \)

Addition \( E = F + G \) \hspace{1cm} \( \dim(\mathcal{T}_F + \mathcal{T}_G) = E \sqcup D(F) \sqcup D(G) \)
Inductive definition of the standard derived term automaton

\[ E \rightsquigarrow \mathcal{T}_E \] by induction on the formation of \( E \)

\[
\begin{align*}
D(F + G) &= D(F) \cup D(G) \\
D(F \cdot G) &= D(F) \cdot G \cup D(G) \\
D(F^*) &= D(F) \cdot F^*
\end{align*}
\]

Property 1 \( \dim \mathcal{T}_E = E \sqcup D(E) \) \quad Property 2 \( \beta : S_E \rightarrow \mathcal{T}_E \) morphism

Base cases

\[
\begin{align*}
\mathcal{T}_0 &= S_0 \\
\mathcal{T}_1 &= S_1 \\
\mathcal{T}_a &= S_a
\end{align*}
\]

Addition \( E = F + G \) \quad \dim(\mathcal{T}_F + \mathcal{T}_G) = E \sqcup D(F) \sqcup D(G)

Natural map \( \varphi : E \sqcup \{D(F) \sqcup D(G)\} \rightarrow E \sqcup \{D(F) \sqcup D(G)\} \)
Inductive definition of the standard derived term automaton

\[ E \leadsto \mathcal{T}_E \] by induction on the formation of \( E \)

\[
\begin{align*}
D(F + G) &= D(F) \cup D(G) \\
D(F \cdot G) &= D(F) \cdot G \cup D(G) \\
D(F^*) &= D(F) \cdot F^*
\end{align*}
\]

Property 1 \quad \text{dim } \mathcal{T}_E = E \sqcup D(E) \quad \text{Property 2} \quad \beta : S_E \to \mathcal{T}_E \text{ morphism}

Base cases \quad \mathcal{T}_0 = S_0 \quad \mathcal{T}_1 = S_1 \quad \mathcal{T}_a = S_a

Addition \quad E = F + G \quad \text{dim}(\mathcal{T}_F + \mathcal{T}_G) = E \sqcup D(F) \sqcup D(G)

Natural map \quad \varphi : E \sqcup \{D(F) \cup D(G)\} \to E \sqcup \{D(F) \cup D(G)\}

Key Proposition 1 \quad \varphi \text{ is a morphism of automata}
Inductive definition of the standard derived term automaton

\[ E \leadsto \mathcal{T}_E \] by induction on the formation of \( E \)

\[ D(F + G) = D(F) \cup D(G) \quad D(F \cdot G) = D(F) \cdot G \cup D(G) \quad D(F^*) = D(F) \cdot F^* \]

Property 1 \( \dim \mathcal{T}_E = E \cup D(E) \) \quad Property 2 \( \beta : S_E \to \mathcal{T}_E \) morphism

Base cases \( \mathcal{T}_0 = S_0 \quad \mathcal{T}_1 = S_1 \quad \mathcal{T}_a = S_a \)

Addition \( E = F + G \quad \mathcal{T}_{F+G} = \varphi(\mathcal{T}_F + \mathcal{T}_G) \)

Natural map \( \varphi : E \cup \{D(F) \cup D(G)\} \to E \cup \{D(F) \cup D(G)\} \)

Key Proposition 1 \( \varphi \) is a morphism of automata
Inductive definition of the standard derived term automaton

\[ E \rightsquigarrow \mathcal{T}_E \] by induction on the formation of \( E \)

\[
D(F + G) = D(F) \cup D(G) \quad D(F \cdot G) = D(F) \cdot G \cup D(G) \quad D(F^*) = D(F) \cdot F^*
\]

Property 1 \( \dim \mathcal{T}_E = E \sqcup D(E) \)  
Property 2 \( \beta : S_E \to \mathcal{T}_E \) morphism

Base cases \[
\mathcal{T}_0 = S_0 \quad \mathcal{T}_1 = S_1 \quad \mathcal{T}_a = S_a
\]

Addition \( E = F + G \)  
\[
\mathcal{T}_{F+G} = \varphi(\mathcal{T}_F + \mathcal{T}_G)
\]

Natural map \( \varphi : E \sqcup \{D(F) \cup D(G)\} \to E \sqcup \{D(F) \cup D(G)\} \)

Key Proposition 1 \( \varphi \) is a morphism of automata

Lemma \( \gamma : S_F \to \mathcal{T}_F \) and \( \delta : S_G \to \mathcal{T}_G \) morphisms  
\[
\gamma \times \delta : (S_F + S_G) \to (\mathcal{T}_F + \mathcal{T}_G) \text{ morphism}
\]
Inductive definition of the standard derived term automaton

\[ E \leadsto \mathcal{T}_E \] by induction on the formation of \( E \)

\[
\begin{align*}
D(F + G) &= D(F) \cup D(G) \\
D(F \cdot G) &= D(F) \cdot G \cup D(G) \\
D(F^*) &= D(F) \cdot F^*
\end{align*}
\]

Property 1 \( \dim \mathcal{T}_E = E \sqcup D(E) \)

Property 2 \( \beta : S_E \rightarrow \mathcal{T}_E \) morphism

Base cases \( \mathcal{T}_0 = S_0 \quad \mathcal{T}_1 = S_1 \quad \mathcal{T}_a = S_a \)

Addition \( E = F + G \quad \mathcal{T}_{F+G} = \varphi(\mathcal{T}_F + \mathcal{T}_G) \)

Product \( E = F \cdot G \)
Inductive definition of the standard derived term automaton

\[ E \leadsto T_E \] by induction on the formation of \( E \)

\[
D(F + G) = D(F) \cup D(G) \quad D(F \cdot G) = D(F) \cdot G \cup D(G) \quad D(F^*) = D(F) \cdot F^*
\]

Property 1 \( \dim T_E = E \sqcup D(E) \) \quad Property 2 \( \beta : S_E \to T_E \) morphism

Base cases \( T_0 = S_0 \) \quad \( T_1 = S_1 \) \quad \( T_a = S_a \)

Addition \( E = F + G \) \quad \( T_{F+G} = \varphi(T_F + T_G) \)

Product \( E = F \cdot G \) \quad \( \dim(T_F \cdot T_G) = E \sqcup D(F) \sqcup D(G) \)

\[
A \cdot B = \left\langle \begin{pmatrix} 0 & J & cK \\ 0 & F & U \cdot K \\ 0 & 0 & G \end{pmatrix}, \begin{pmatrix} c \, d \\ Ud \\ V \end{pmatrix} \rightangle
\]
Inductive definition of the standard derived term automaton

\[ E \leadsto \mathcal{T}_E \] by induction on the formation of \( E \)

\[ \mathcal{D}(F + G) = \mathcal{D}(F) \cup \mathcal{D}(G) \quad \mathcal{D}(F \cdot G) = \mathcal{D}(F) \cdot G \cup \mathcal{D}(G) \quad \mathcal{D}(F^*) = \mathcal{D}(F) \cdot F^* \]

Property 1 \( \dim \mathcal{T}_E = E \sqcup \mathcal{D}(E) \) \quad Property 2 \( \beta : S_E \rightarrow \mathcal{T}_E \) morphism

Base cases \( \mathcal{T}_0 = S_0 \quad \mathcal{T}_1 = S_1 \quad \mathcal{T}_a = S_a \)

Addition \( E = F + G \quad \mathcal{T}_{F+G} = \varphi(\mathcal{T}_F + \mathcal{T}_G) \)

Product \( E = F \cdot G \quad \dim(\mathcal{T}_F \cdot \mathcal{T}_G) = E \sqcup \mathcal{D}(F) \sqcup \mathcal{D}(G) \)

Natural map \( \psi : E \sqcup \{\mathcal{D}(F) \sqcup \mathcal{D}(G)\} \rightarrow E \sqcup \{\mathcal{D}(F) \cdot G \sqcup \mathcal{D}(G)\} \)
Inductive definition of the standard derived term automaton

\[ E \rightsquigarrow \mathcal{T}_E \] by induction on the formation of \( E \)

\[
D(F + G) = D(F) \cup D(G) \quad D(F \cdot G) = D(F) \cdot G \cup D(G) \quad D(F^*) = D(F) \cdot F^*
\]

Property 1  \( \dim \mathcal{T}_E = E \sqcup D(E) \)  
Property 2  \( \beta : S_E \rightarrow \mathcal{T}_E \) morphism

Base cases  
\[
\mathcal{T}_0 = S_0 \quad \mathcal{T}_1 = S_1 \quad \mathcal{T}_a = S_a
\]

Addition  
\[
E = F + G \quad \mathcal{T}_{F+G} = \varphi(\mathcal{T}_F + \mathcal{T}_G)
\]

Product  
\[
E = F \cdot G \quad \dim(\mathcal{T}_F \cdot \mathcal{T}_G) = E \sqcup D(F) \sqcup D(G)
\]

Natural map  \( \psi : E \sqcup \{D(F) \sqcup D(G)\} \rightarrow E \sqcup \{D(F) \cdot G \sqcup D(G)\} \)

Key Proposition 2  \( \psi \text{ is a morphism of automata} \)
Inductive definition of the standard derived term automaton

\[ E \leadsto \mathcal{T}_E \] by induction on the formation of \( E \)

\[
\begin{align*}
D(F + G) &= D(F) \cup D(G) \\
D(F \cdot G) &= D(F) \cdot G \cup D(G) \\
D(F^*) &= D(F) \cdot F^*
\end{align*}
\]

Property 1 \( \dim \mathcal{T}_E = E \sqcup D(E) \) \quad Property 2 \( \beta : S_E \to \mathcal{T}_E \) morphism

Base cases \( \mathcal{T}_0 = S_0 \) \quad \( \mathcal{T}_1 = S_1 \) \quad \( \mathcal{T}_a = S_a \)

Addition \( E = F + G \) \quad \( \mathcal{T}_{F+G} = \varphi(\mathcal{T}_F + \mathcal{T}_G) \)

Product \( E = F \cdot G \) \quad \( \mathcal{T}_{F \cdot G} = \psi(\mathcal{T}_F \cdot \mathcal{T}_G) \)

Natural map \( \psi : E \sqcup \{D(F) \sqcup D(G)\} \to E \sqcup \{D(F) \cdot G \cup D(G)\} \)

Key Proposition 2 \( \psi \) is a morphism of automata
Inductive definition of the standard derived term automaton

\[ E \rightsquigarrow \mathcal{T}_E \] by induction on the formation of \( E \)

\[
\begin{align*}
D(F + G) &= D(F) \cup D(G) \\
D(F \cdot G) &= D(F) \cdot G \cup D(G) \\
D(F^*) &= D(F) \cdot F^*
\end{align*}
\]

Property 1 \( \dim \mathcal{T}_E = E \sqcup D(E) \) \hspace{1cm} Property 2 \( \beta : S_E \to \mathcal{T}_E \) morphism

Base cases

\[
\begin{align*}
\mathcal{T}_0 &= S_0 \\
\mathcal{T}_1 &= S_1 \\
\mathcal{T}_a &= S_a
\end{align*}
\]

Addition \( E = F + G \) \hspace{1cm} \( \mathcal{T}_{F+G} = \varphi(\mathcal{T}_F + \mathcal{T}_G) \)

Product \( E = F \cdot G \) \hspace{1cm} \( \mathcal{T}_{F \cdot G} = \psi(\mathcal{T}_F \cdot \mathcal{T}_G) \)

Natural map \( \psi : E \sqcup \{D(F) \sqcup D(G)\} \to E \sqcup \{D(F) \cdot G \sqcup D(G)\} \)

Key Proposition 2 \( \psi \) is a morphism of automata

Lemma \( \gamma : S_F \to \mathcal{T}_F \) and \( \delta : S_G \to \mathcal{T}_G \) morphisms \( \implies \gamma \times \delta : (S_F \cdot S_G) \to (\mathcal{T}_F \cdot \mathcal{T}_G) \) morphism
Inductive definition of the standard derived term automaton

\[ E \leadsto \mathcal{T}_E \] by induction on the formation of \( E \)

\[
\begin{align*}
D(F + G) &= D(F) \cup D(G) \\
D(F \cdot G) &= D(F) \cdot G \cup D(G) \\
D(F^*) &= D(F) \cdot F^*
\end{align*}
\]

Property 1 \( \dim \mathcal{T}_E = E \sqcup D(E) \)  
Property 2 \( \beta : S_E \to \mathcal{T}_E \) morphism

Base cases  
\[
\begin{align*}
\mathcal{T}_0 &= S_0 \\
\mathcal{T}_1 &= S_1 \\
\mathcal{T}_a &= S_a
\end{align*}
\]

Addition  
\( E = F + G \)  
\( \mathcal{T}_{F+G} = \varphi(\mathcal{T}_F + \mathcal{T}_G) \)

Product  
\( E = F \cdot G \)  
\( \mathcal{T}_{F \cdot G} = \psi(\mathcal{T}_F \cdot \mathcal{T}_G) \)

Star  
\( E = F^* \)  
\( \mathcal{T}_{F^*} = \mathcal{T}_F^* \)
Inductive definition of the standard derived term automaton

$E \leadsto \mathcal{T}_E$ by induction on the formation of $E$

$D(F + G) = D(F) \cup D(G)$  $D(F \cdot G) = D(F) \cdot G \cup D(G)$  $D(F^*) = D(F) \cdot F^*$

Property 1  $\dim \mathcal{T}_E = E \sqcup D(E)$  Property 2  $\beta : S_E \rightarrow \mathcal{T}_E$ morphism

Base cases

$\mathcal{T}_0 = S_0$  $\mathcal{T}_1 = S_1$  $\mathcal{T}_a = S_a$

Addition

$E = F + G$  $\mathcal{T}_{F+G} = \varphi(\mathcal{T}_F + \mathcal{T}_G)$

Product

$E = F \cdot G$  $\mathcal{T}_{F \cdot G} = \psi(\mathcal{T}_F \cdot \mathcal{T}_G)$

Star

$E = F^*$  $\mathcal{T}_{F^*} = \mathcal{T}_F^*$

Lemma  $\gamma : S_F \rightarrow \mathcal{T}_F$ morphism  $\implies$  $\gamma : S_F^* \rightarrow \mathcal{T}_F^*$ morphism
Inductive definition of the standard derived term automaton

\[ E \leadsto \mathcal{T}_E \] by induction on the formation of \( E \)

\[ D(F + G) = D(F) \cup D(G) \]
\[ D(F \cdot G) = D(F) \cdot G \cup D(G) \]
\[ D(F^*) = D(F) \cdot F^* \]

Property 1 \( \dim \mathcal{T}_E = E \sqcup D(E) \)

Property 2 \( \beta : S_E \rightarrow \mathcal{T}_E \) morphism

Base cases
\[ \mathcal{T}_0 = S_0 \]
\[ \mathcal{T}_1 = S_1 \]
\[ \mathcal{T}_a = S_a \]

Addition \( E = F + G \)
\[ \mathcal{T}_{F+G} = \varphi(\mathcal{T}_F + \mathcal{T}_G) \]

Product \( E = F \cdot G \)
\[ \mathcal{T}_{F \cdot G} = \psi(\mathcal{T}_F \cdot \mathcal{T}_G) \]

Star \( E = F^* \)
\[ \mathcal{T}_{F^*} = \mathcal{T}_F^* \]
Inductive definition of the standard derived term automaton

\[ E \leadsto T_E \] by induction on the formation of \( E \)

\[
\begin{align*}
D(F + G) &= D(F) \cup D(G) \\
D(F \cdot G) &= D(F) \cdot G \cup D(G) \\
D(F^*) &= D(F) \cdot F^*
\end{align*}
\]

Property 1 \( \dim T_E = E \sqcup D(E) \) \hspace{1cm} Property 2 \( \beta : S_E \to T_E \) morphism

Base cases \( T_0 = S_0 \), \( T_1 = S_1 \), \( T_a = S_a \)

Addition \( E = F + G \) \( T_{F+G} = \varphi(T_F + T_G) \)

Product \( E = F \cdot G \) \( T_{F \cdot G} = \psi(T_F \cdot T_G) \)

Star \( E = F^* \) \( T_{F^*} = T_F^* \)

Derived term automaton
Inductive definition of the standard derived term automaton

\[ E \leadsto \mathcal{T}_E \] by induction on the formation of \( E \)

\[
\begin{align*}
D(F + G) &= D(F) \cup D(G) \\
D(F \cdot G) &= D(F) \cdot G \cup D(G) \\
D(F^*) &= D(F) \cdot F^*
\end{align*}
\]

Property 1 \( \dim \mathcal{T}_E = E \sqcup D(E) \)  \quad Property 2 \( \beta : S_E \to \mathcal{T}_E \) morphism

Base cases

\[
\begin{align*}
\mathcal{T}_0 &= S_0 \\
\mathcal{T}_1 &= S_1 \\
\mathcal{T}_a &= S_a
\end{align*}
\]

Addition \( E = F + G \) \quad \mathcal{T}_{F+G} = \varphi(\mathcal{T}_F + \mathcal{T}_G)

Product \( E = F \cdot G \) \quad \mathcal{T}_{F \cdot G} = \psi(\mathcal{T}_F \cdot \mathcal{T}_G)

Star \( E = F^* \) \quad \mathcal{T}_{F^*} = \mathcal{T}_F^*

Derived term automaton \quad \text{Natural map} \quad \omega : E \sqcup D(E) \to E \sqcup D(E)
Inductive definition of the standard derived term automaton

\[ E \leadsto T_E \] by induction on the formation of \( E \)

\[
\begin{align*}
D(F + G) &= D(F) \cup D(G) \\
D(F \cdot G) &= D(F) \cdot G \cup D(G) \\
D(F^*) &= D(F) \cdot F^*
\end{align*}
\]

Property 1 \( \dim T_E = E \sqcup D(E) \)  
Property 2 \( \beta : S_E \to T_E \) morphism

Base cases \( T_0 = S_0 \quad T_1 = S_1 \quad T_a = S_a \)

Addition \( E = F + G \) \( T_{F+G} = \varphi(T_F + T_G) \)

Product \( E = F \cdot G \) \( T_{F \cdot G} = \psi(T_F \cdot T_G) \)

Star \( E = F^* \) \( T_{F^*} = T_F^* \)

Derived term automaton \( \)  

Natural map \( \omega : E \sqcup D(E) \to E \sqcup D(E) \)

Key Proposition 3 \( \omega \) is a morphism of automata
Inductive definition of the standard derived term automaton

\[ E \leadsto \mathcal{T}_E \] by induction on the formation of \( E \)

\[
\begin{align*}
D(F + G) &= D(F) \cup D(G) \\
D(F \cdot G) &= D(F) \cdot G \cup D(G) \\
D(F^*) &= D(F) \cdot F^*
\end{align*}
\]

Property 1 \( \dim \mathcal{T}_E = E \sqcup D(E) \)  
Property 2 \( \beta : S_E \to \mathcal{T}_E \) morphism

Base cases \( \mathcal{T}_0 = S_0 \), \( \mathcal{T}_1 = S_1 \), \( \mathcal{T}_a = S_a \)

Addition \( E = F + G \) \( \mathcal{T}_{F+G} = \varphi(\mathcal{T}_F + \mathcal{T}_G) \)

Product \( E = F \cdot G \) \( \mathcal{T}_{F \cdot G} = \psi(\mathcal{T}_F \cdot \mathcal{T}_G) \)

Star \( E = F^* \) \( \mathcal{T}_{F^*} = \mathcal{T}_F^* \)

Derived term automaton \( \mathcal{T}_E \)  
Natural map \( \omega : E \sqcup D(E) \to E \cup D(E) \)

Key Proposition 3 \( \omega \) is a morphism of automata

Theorem \( \mathcal{D}_E = \omega(\mathcal{T}_E) \) is the derived term automaton of \( E \)
- Standard automaton of $E$ position, Glushkov
- Derived term automaton of $E$ Brzozowski–Antimirov

\[ \Delta_s \]

\[ \Delta_d \]

\[ \text{Rat}E \ M \quad \text{E} \quad \Delta \]

\[ \text{Aut} \ M \quad \text{D} \quad \Delta \]

\[ |S_E| = |E| \quad \dim S_E = \ell(E) + 1 \]

\[ |D_E| = |E| \quad \dim D_E \leq \ell(E) + 1 \]

\[ \alpha: S_E \rightarrow D_E \quad \text{morphism of automata.} \]