Compression by deterministic pushdown automata

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2022 – ANR Delta

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Outline

Normality

Characterization by non-compressibility

Non-deterministic pushdown transducers

Deterministic pushdown transducers

Outline

Normality (Borel 1909)

The number of occurrences of a word w in a word u is

$$\mathrm{occ}(u, w) = \#\{i : u[i{:}i + |w| - 1] = w\}$$

A sequence $x \in A^{\mathbb{N}}$ normal if for each $w \in A^*$,

$$\lim_{n\to\infty} \frac{\mathrm{occ}(x[1{:}n], w}{n} = \frac{1}{(\#A)^{|w|}}.$$

For $A = \{0, 1\}$, this means

- the frequencies in x of the 2 digits 0 and 1 are $1/2$,
- the frequencies in x of the 4 words $00, 01, 10, 11$ are $1/4$,
- the frequencies in x of the 8 words $000, 001, \ldots, 111$ are $1/8$,
- $\ldots$

Examples

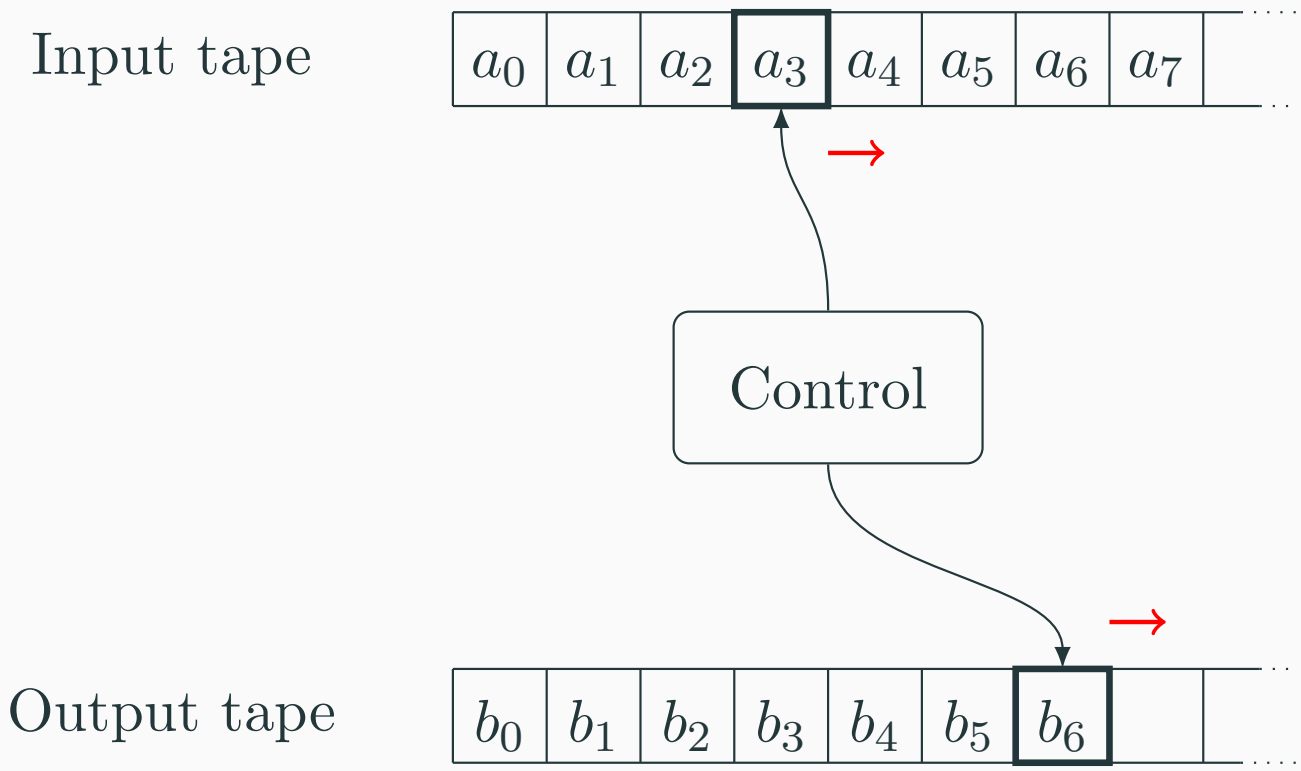
Theorem (Borel 1909)
Almost all real numbers are normal, that is, the measure of the set of normal numbers in $[0, 1)$ *is* 1.

Examples

- the Champernowne sequence $01\,00011011\,000001010\cdots$,
- the sequence of primes in binary $11011101111110111101\cdots$.
- It is not known whether all digits occur infinitely many times in the base 3 expansion of $\sqrt{2}$.

Outline

Transducers



Transducers as compressors

A transducer $\mathcal{T}$ *compresses* a sequence $x = a_1 a_2 a_3 \cdots$ if

- the function (relation) realized by $\mathcal{T}$ is one-to-one,
- there is an accepting run with input x

$$C_0 \xrightarrow{a_1|v_1} C_1 \xrightarrow{a_2|v_2} C_2 \xrightarrow{a_3|v_3} C_3 \cdots$$

such that

$$\liminf_{n\to\infty} \frac{|v_1 v_2 \cdots v_n| \log \#B}{|a_1 a_2 \cdots a_n| \log \#A} < 1.$$

Characterization of normal sequences

Theorem (Many people)
A sequence is normal if and only if it cannot be compressed by deterministic one-to-one finite state transducer.

- Schnorr and Stimm (1971)
  non-normality $\Leftrightarrow$ finite-state martingale success
- Dai, Lathrop, Lutz and Mayordomo (2004)
  compressibility $\Leftrightarrow$ finite-state martingale success
  normality $\Rightarrow$ no martingale success
- Bourke, Hitchcock and Vinodchandran (2005)
  non-normality $\Rightarrow$ martingale success
- Becher and Heiber (2013)
  non-normality $\Leftrightarrow$ compressibility (direct)

Summary of the known results

	det	non-det	non-rt
finite-state	N	N	N
1 counter	N	N	N
$\geqslant 2$ counters	N	N	T
1 stack	?	C	C
1 stack + 1 counter	C	C	T

where

N means *cannot compress normal sequences*

C means *can compress some normal sequence*

T means *is Turing complete* and thus can compress.

Outline

Non-deterministic pushdown transducers

Let A be an alphabet of even cardinality. Let w_n be the concatenation in some order of all words of length n.

$$x = w_1 \tilde{w}_1 w_2 \tilde{w}_2 w_3 \tilde{w}_3 \cdots$$

where $\tilde{w}$ is the reverse of w. For $A = \{0, 1\}$, it gives

$$x = 0\,1\,1\,0\,00\,01\,10\,11\,11\,01\,10\,00\,000\,001\,010\,\cdots$$

The run is an alternation of pushing and popping phases.

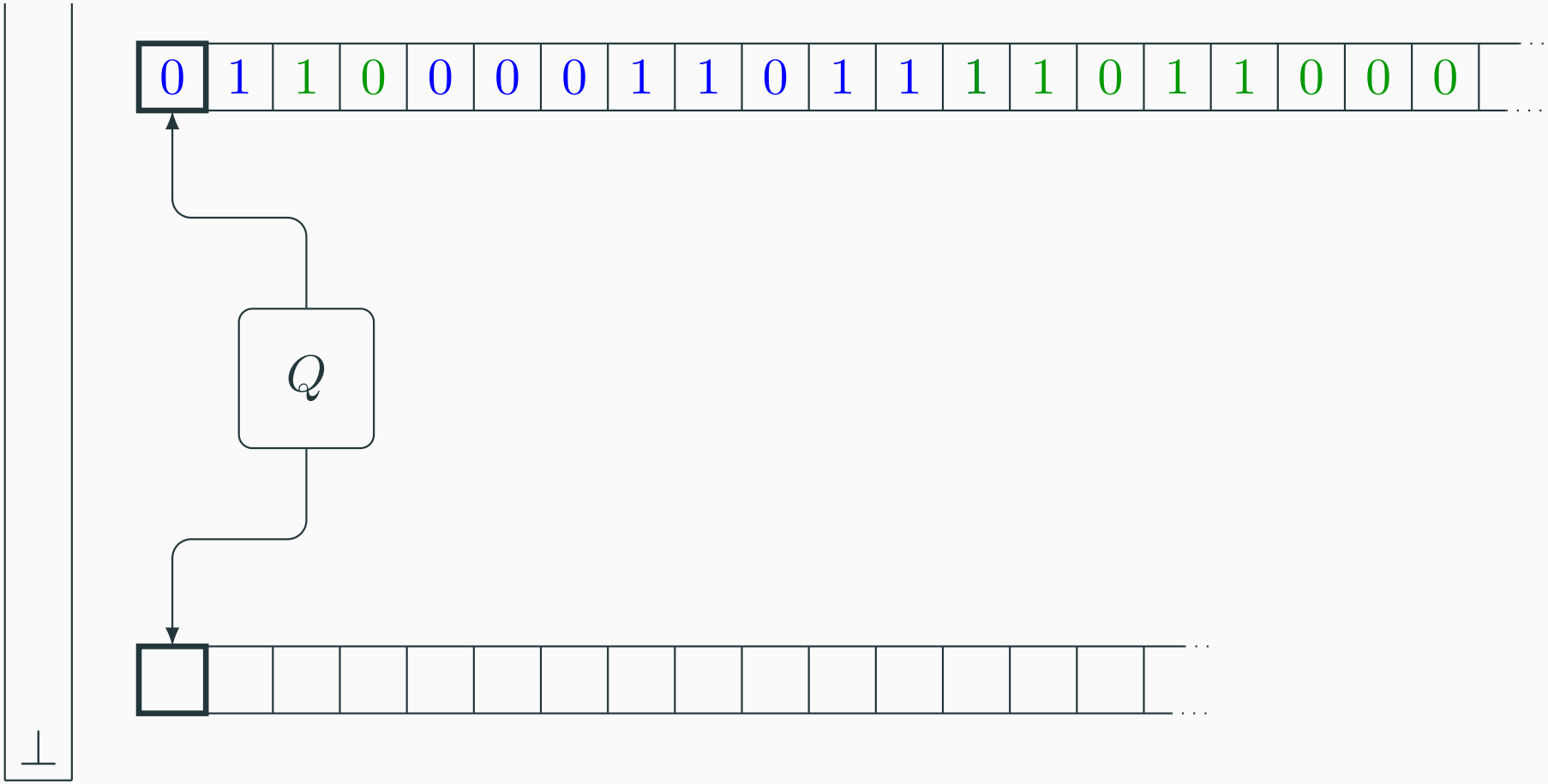
- In the pushing phase, each input symbol is pushed and output.
- In the popping phase, each input symbol must be equal to the stack top symbol and this top symbol is popped.
  A symbol $\square$ is output every two popped symbols.

Non-deterministic pushdown transducers

$$x = w_1\tilde{w}_1w_2\tilde{w}_2w_3\tilde{w}_3\cdots$$

- Each word w_n is read during a pushing phase.
- At the end of w_n, the transducer switches non-deterministically to a popping phase to read $\tilde{w}_n$.
- Each word $\tilde{w}_n$ is read during a popping phase.
- At the end of $\tilde{w}_n$, the stack is empty and the transducer switches back to a pushing phase to read w_{n+1}.
- While reading $w_n\tilde{w}_n$, the transducer outputs $w_n\square^{|w_n|/2}$.

Let it run



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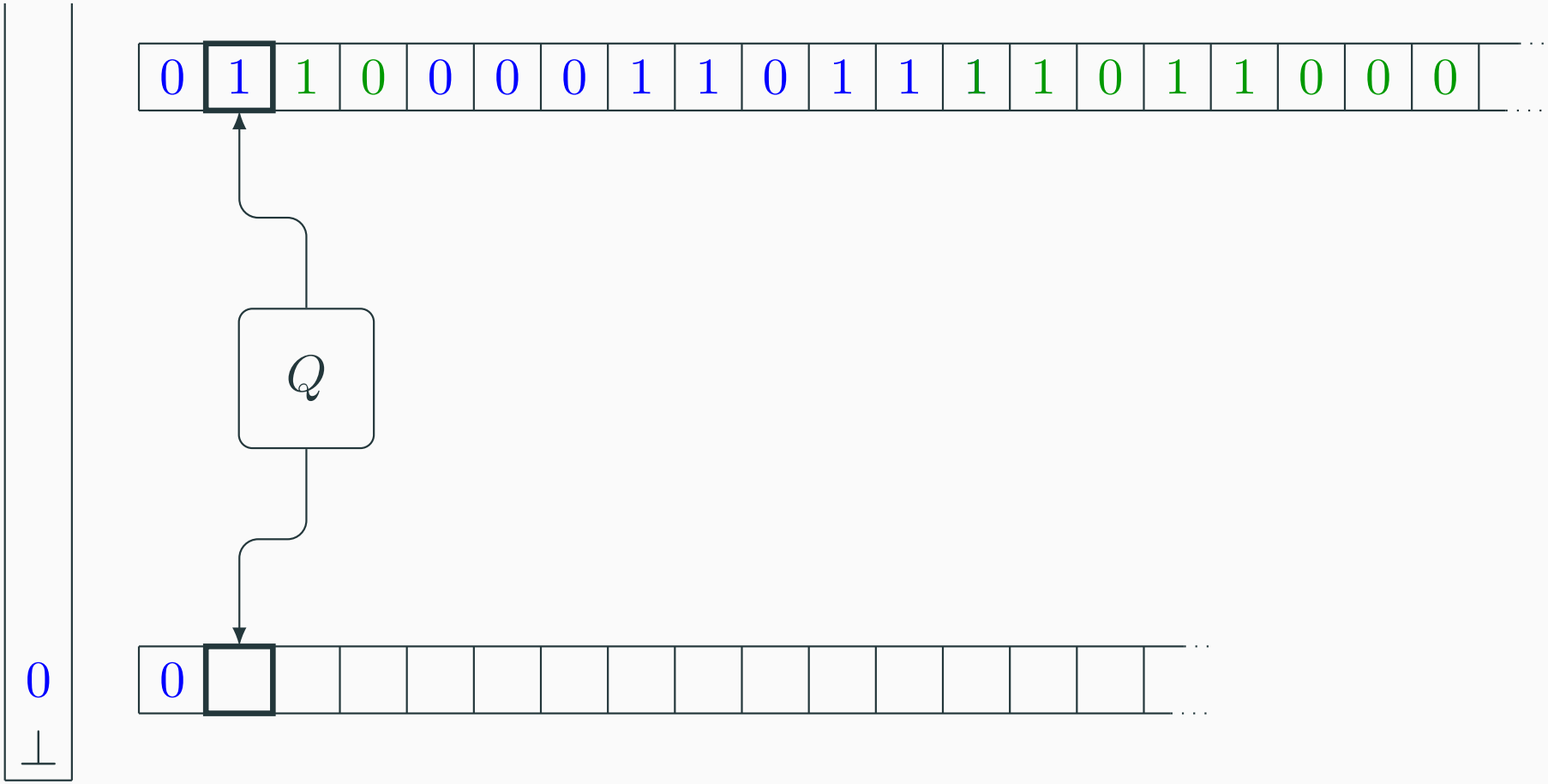

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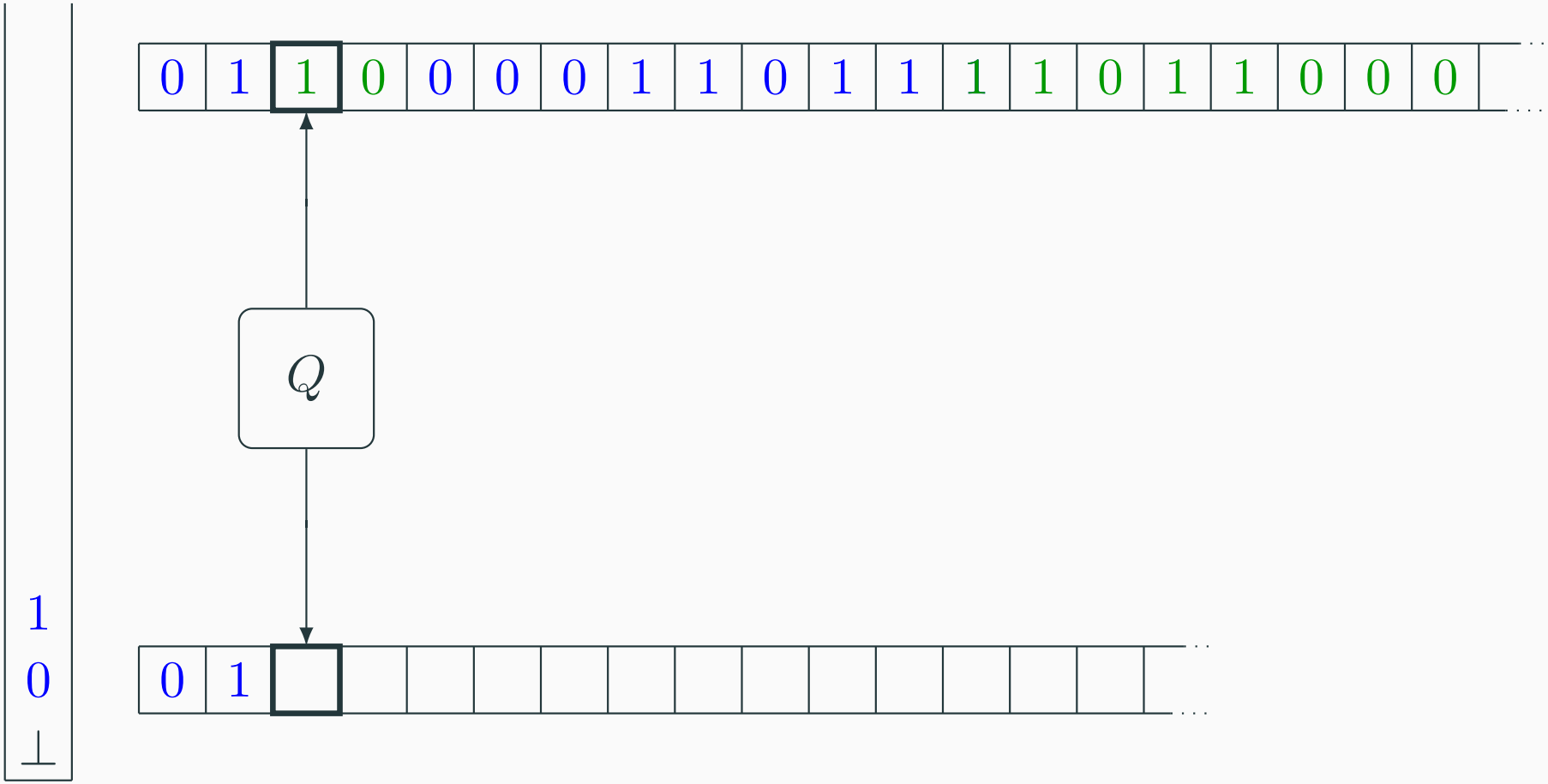

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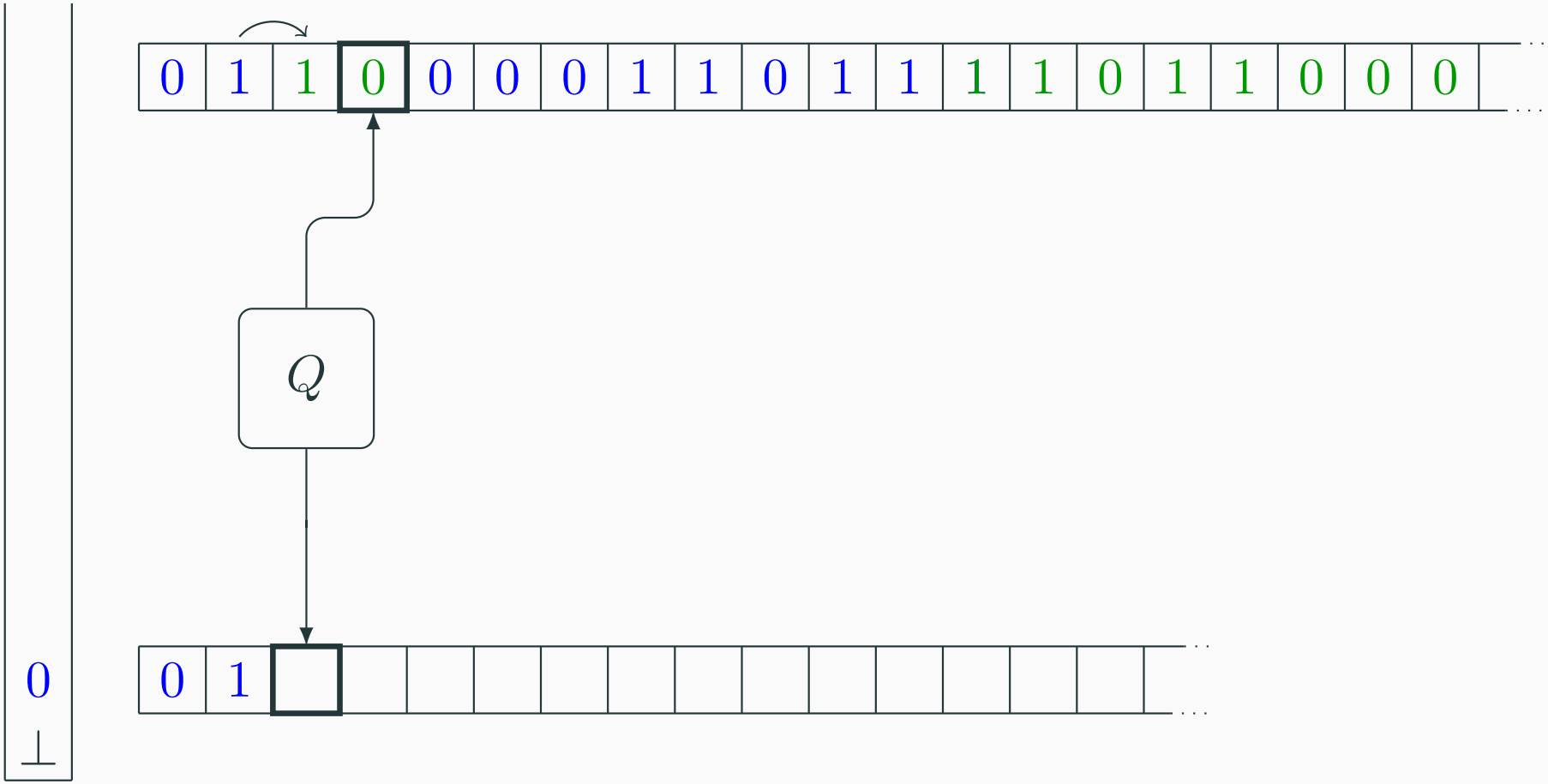

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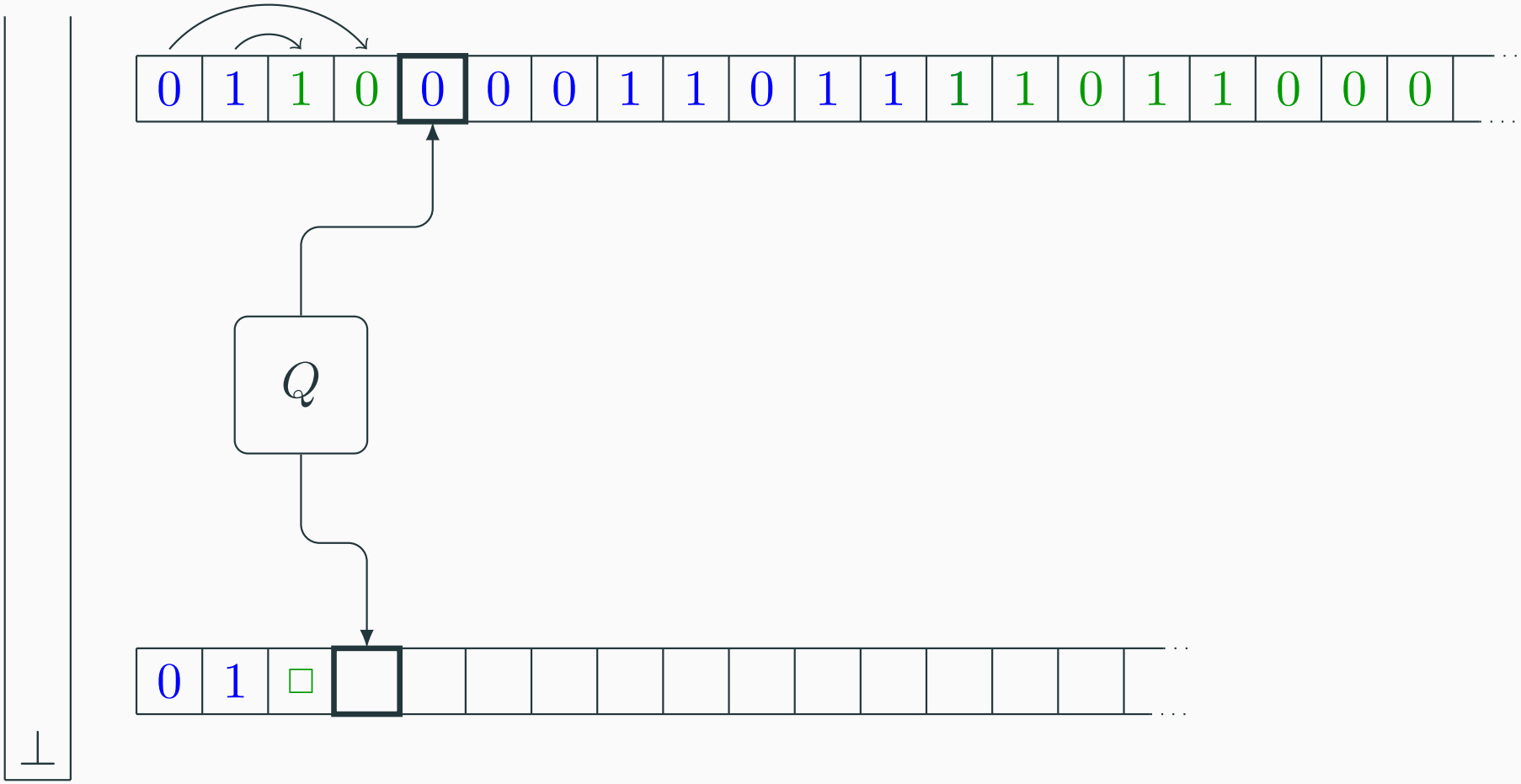

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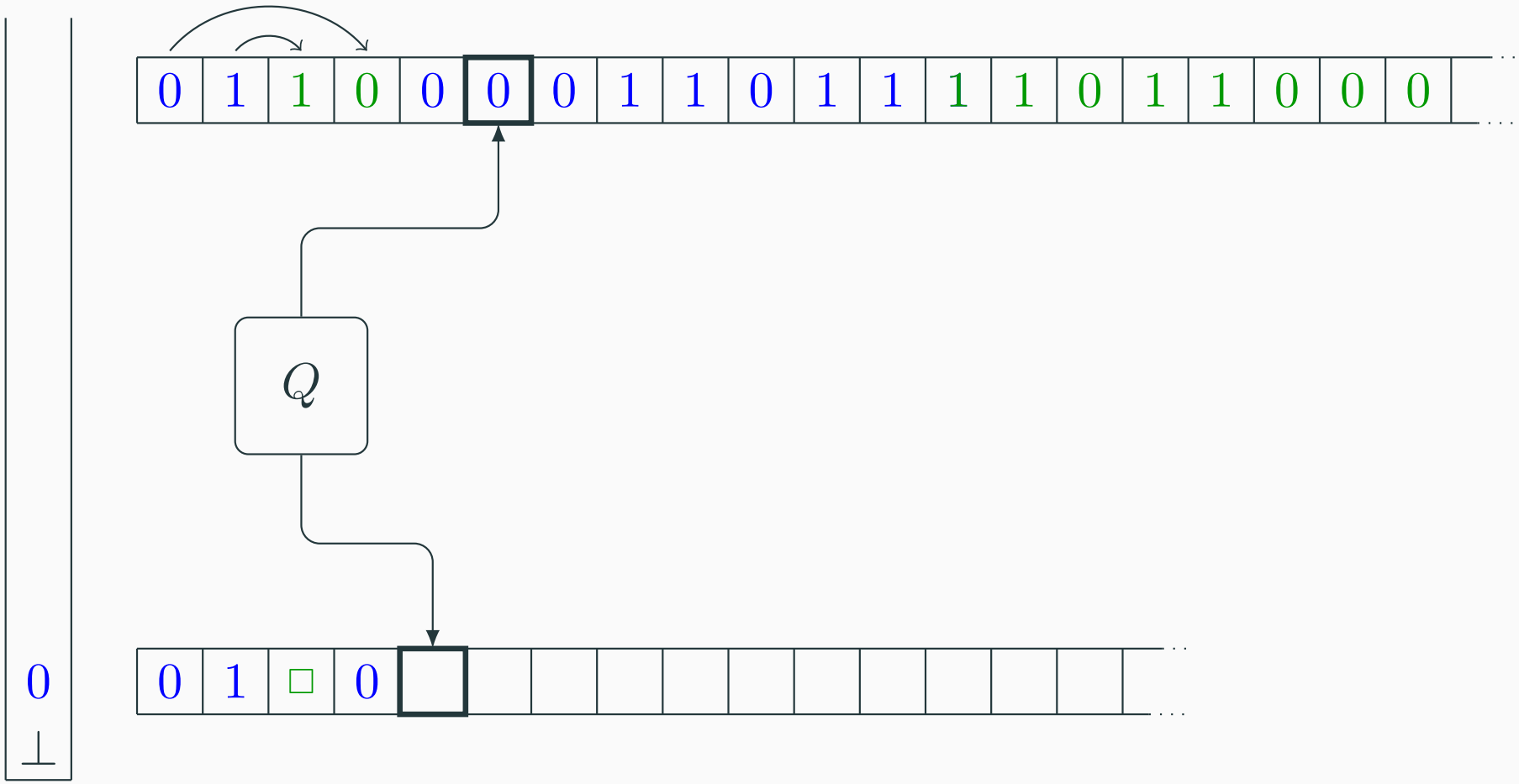

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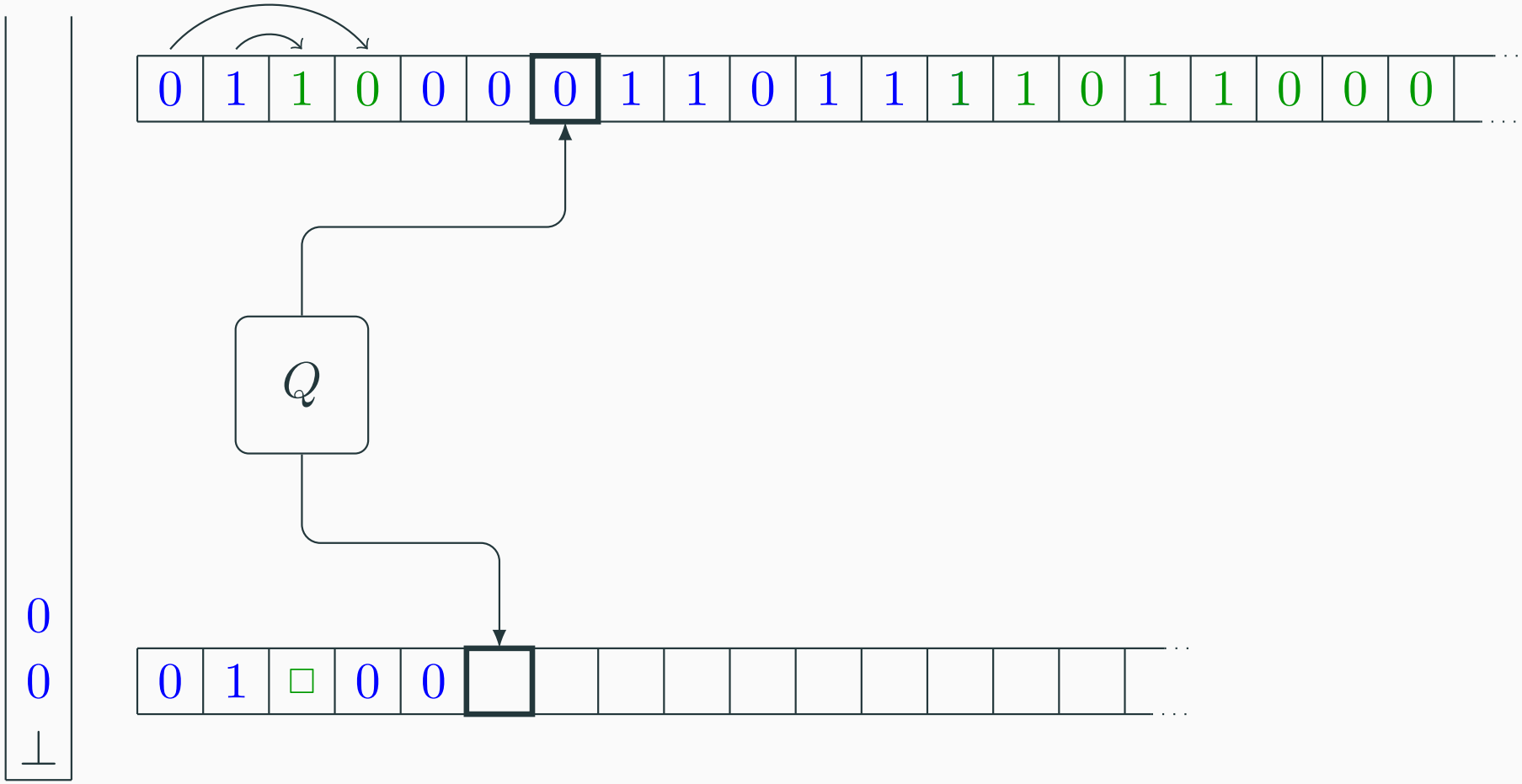

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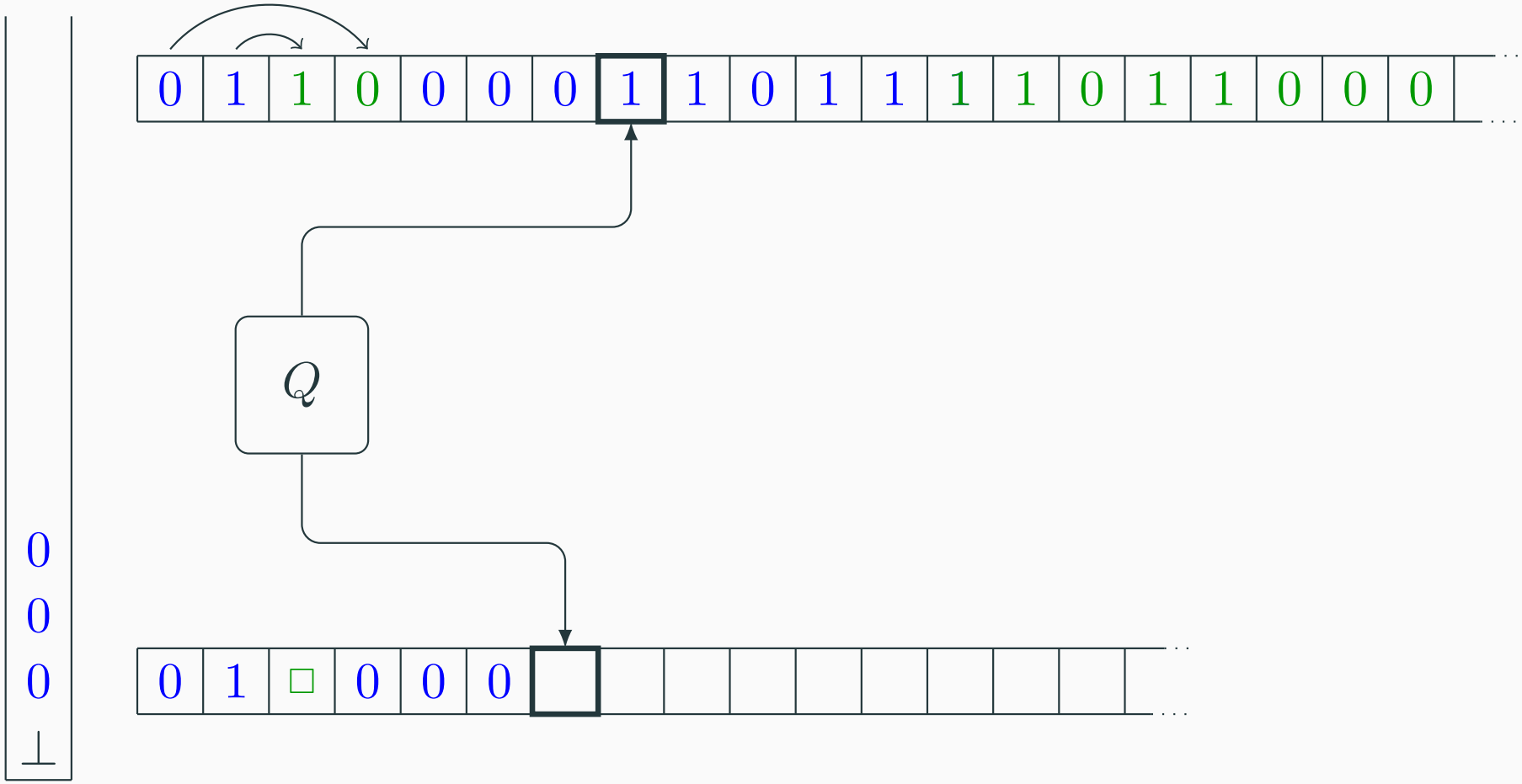

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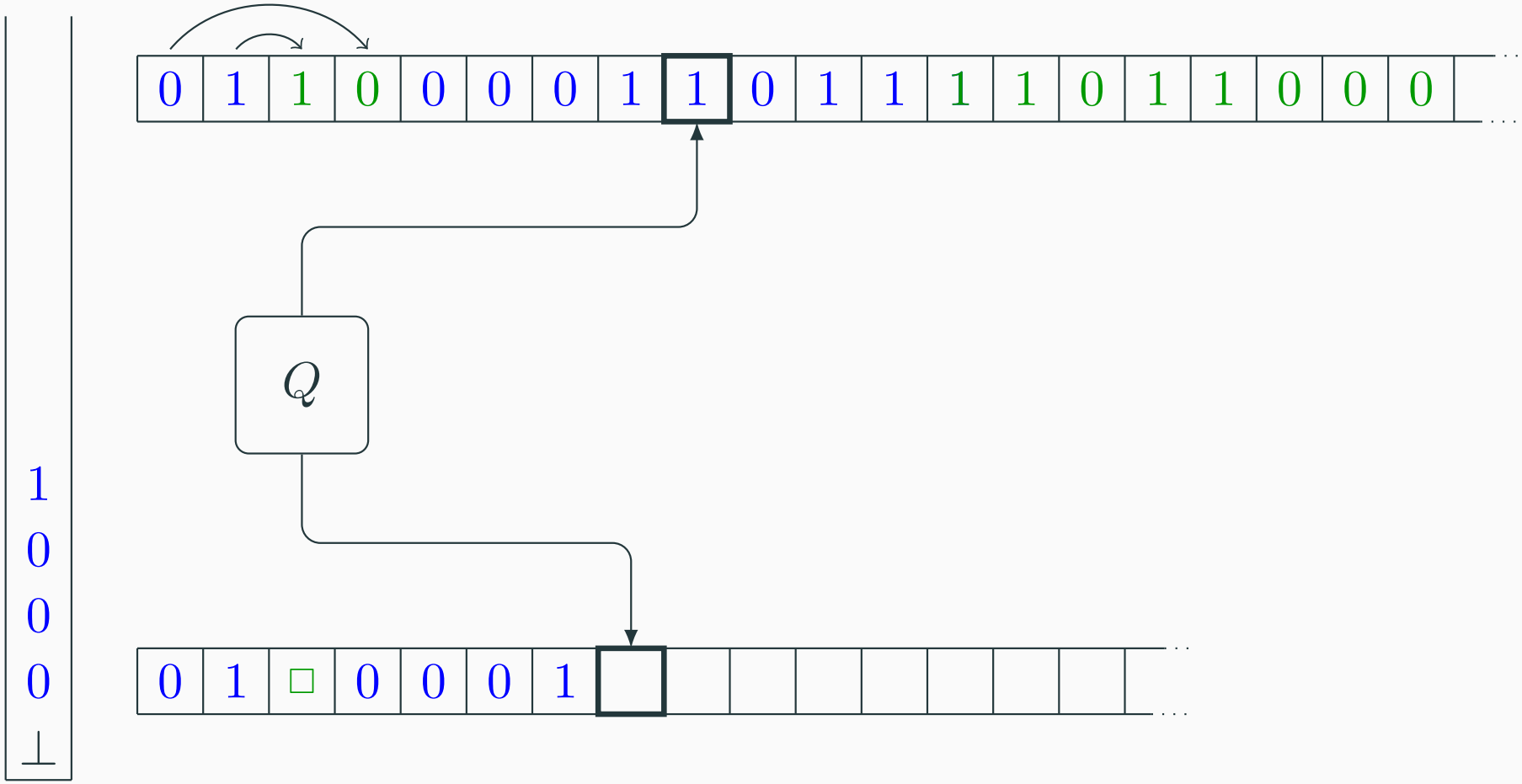

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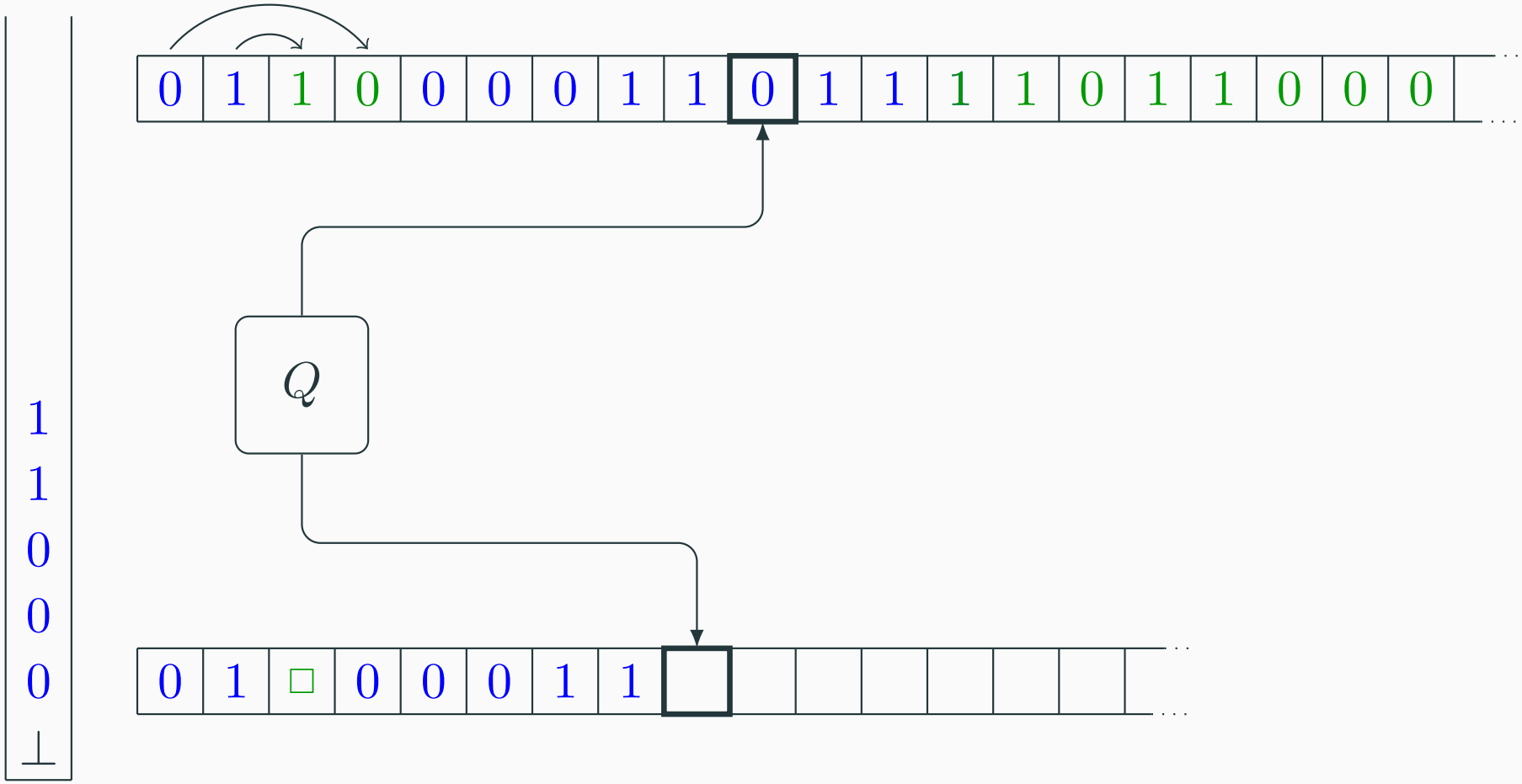

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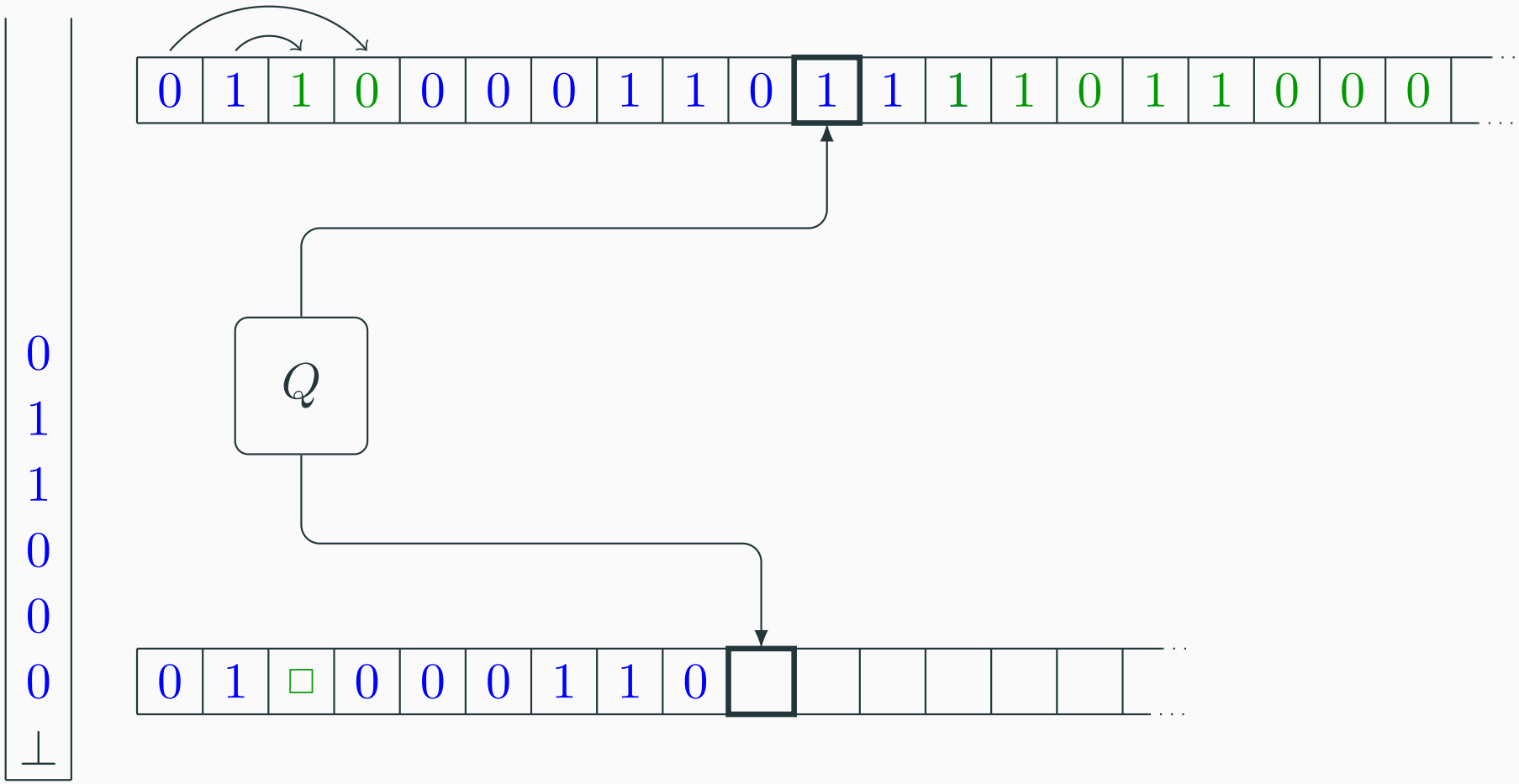

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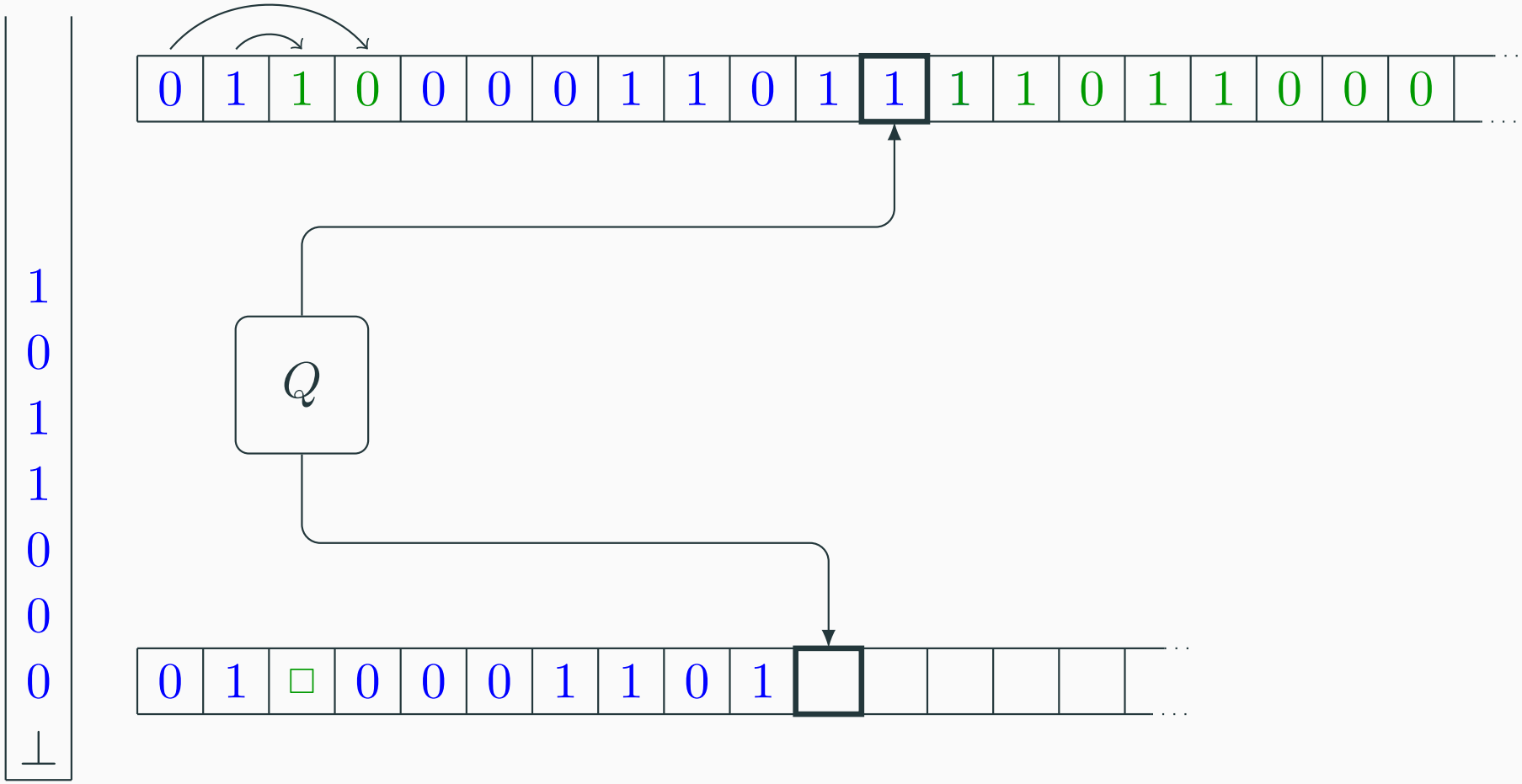

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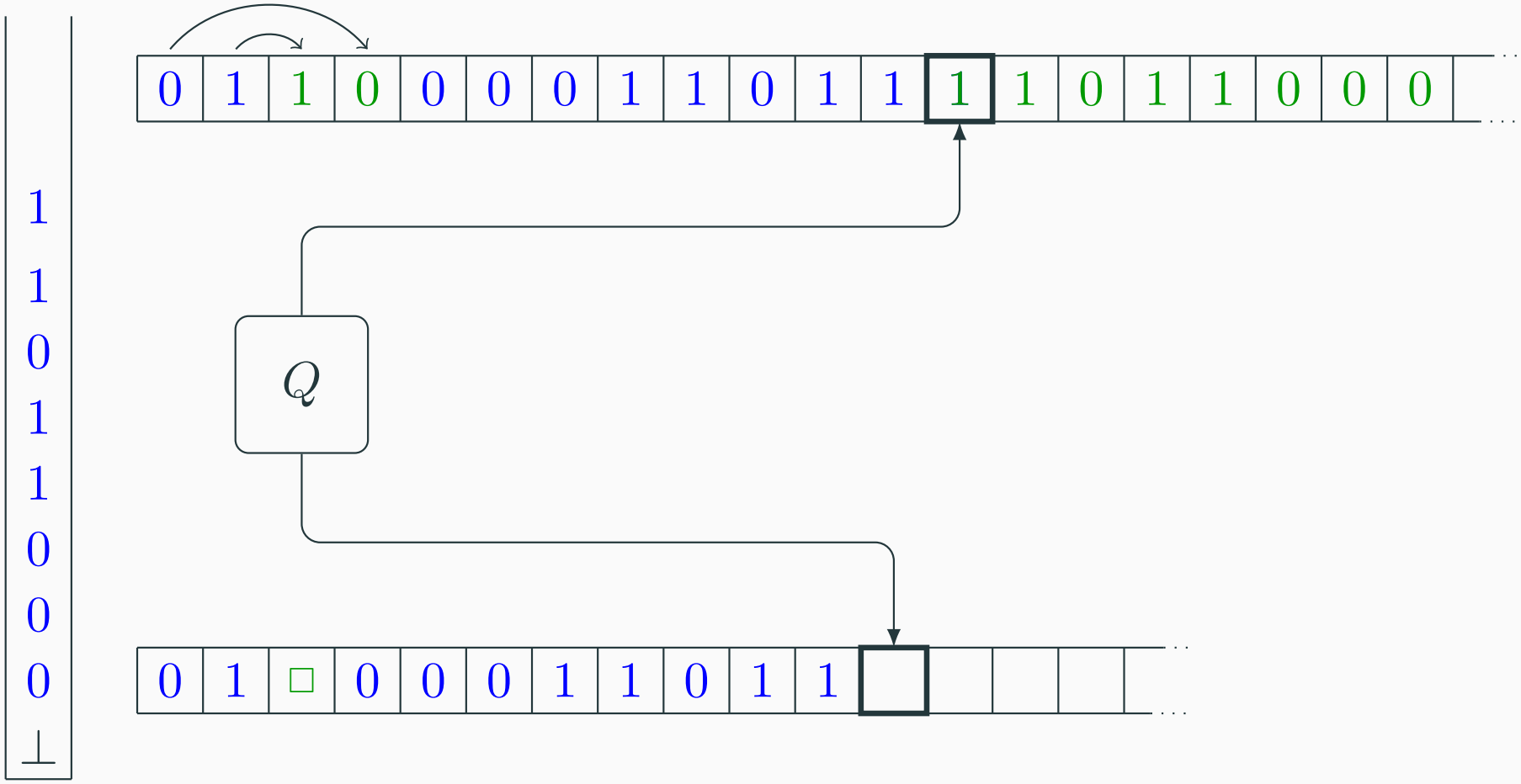

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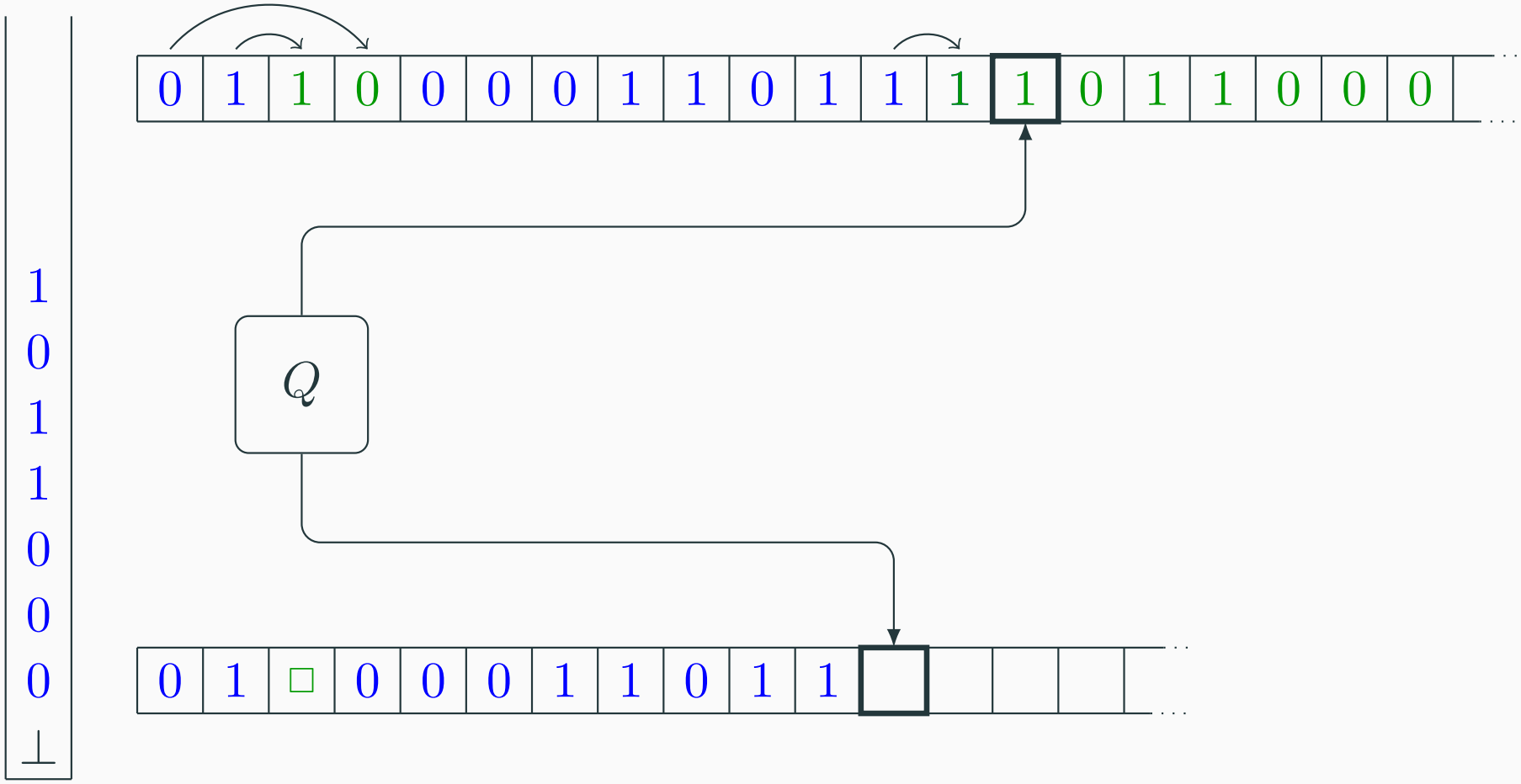

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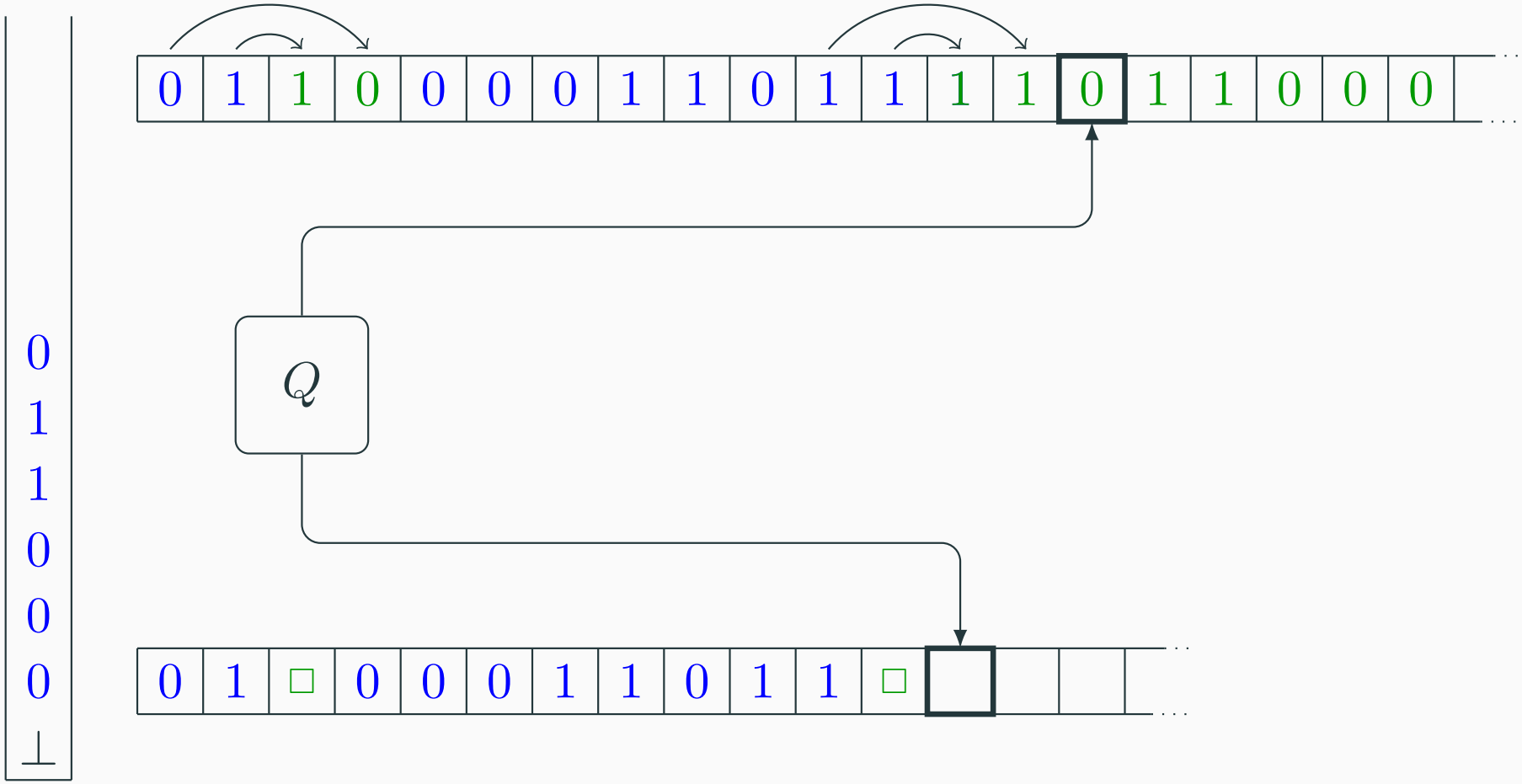

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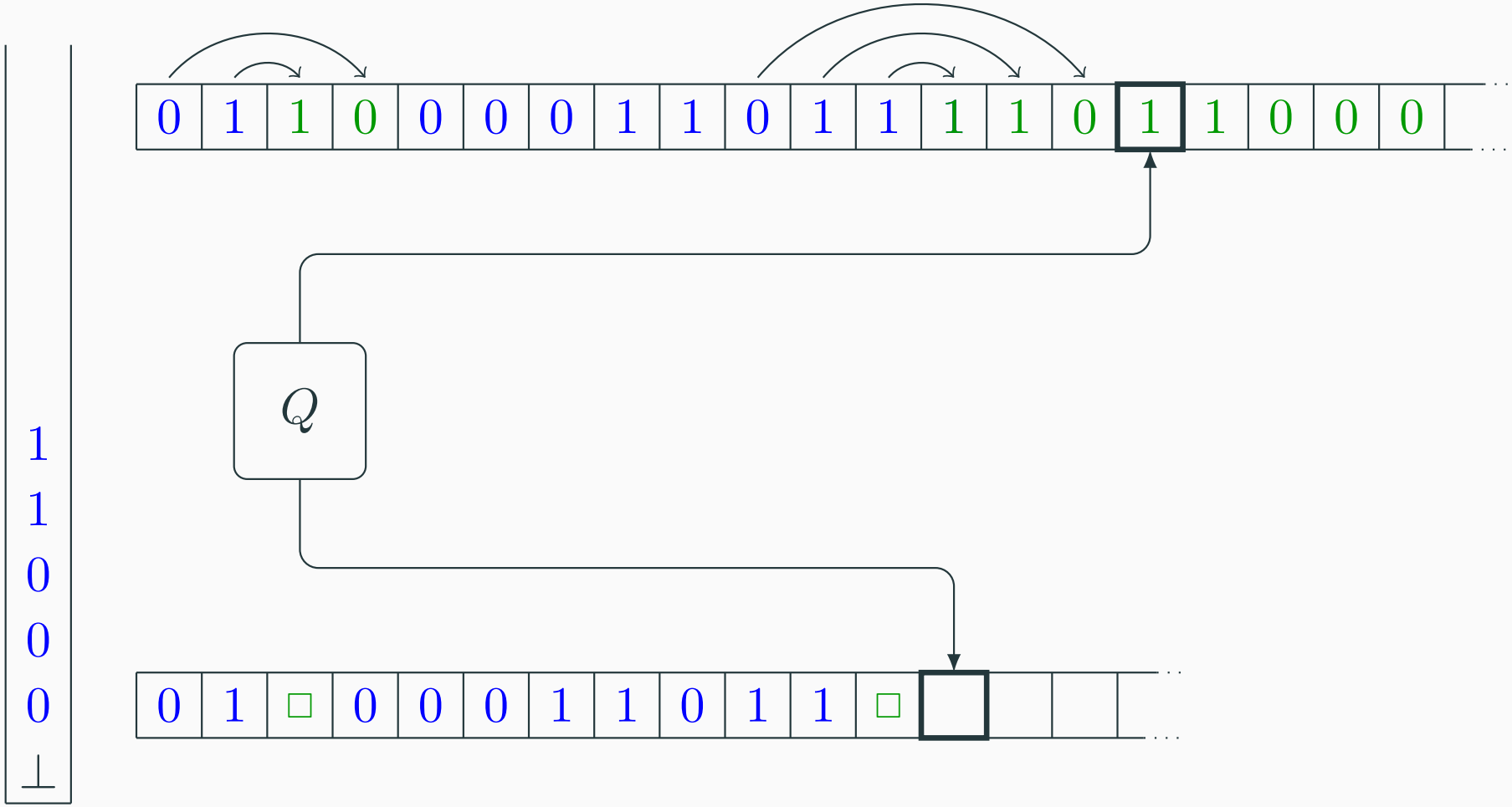

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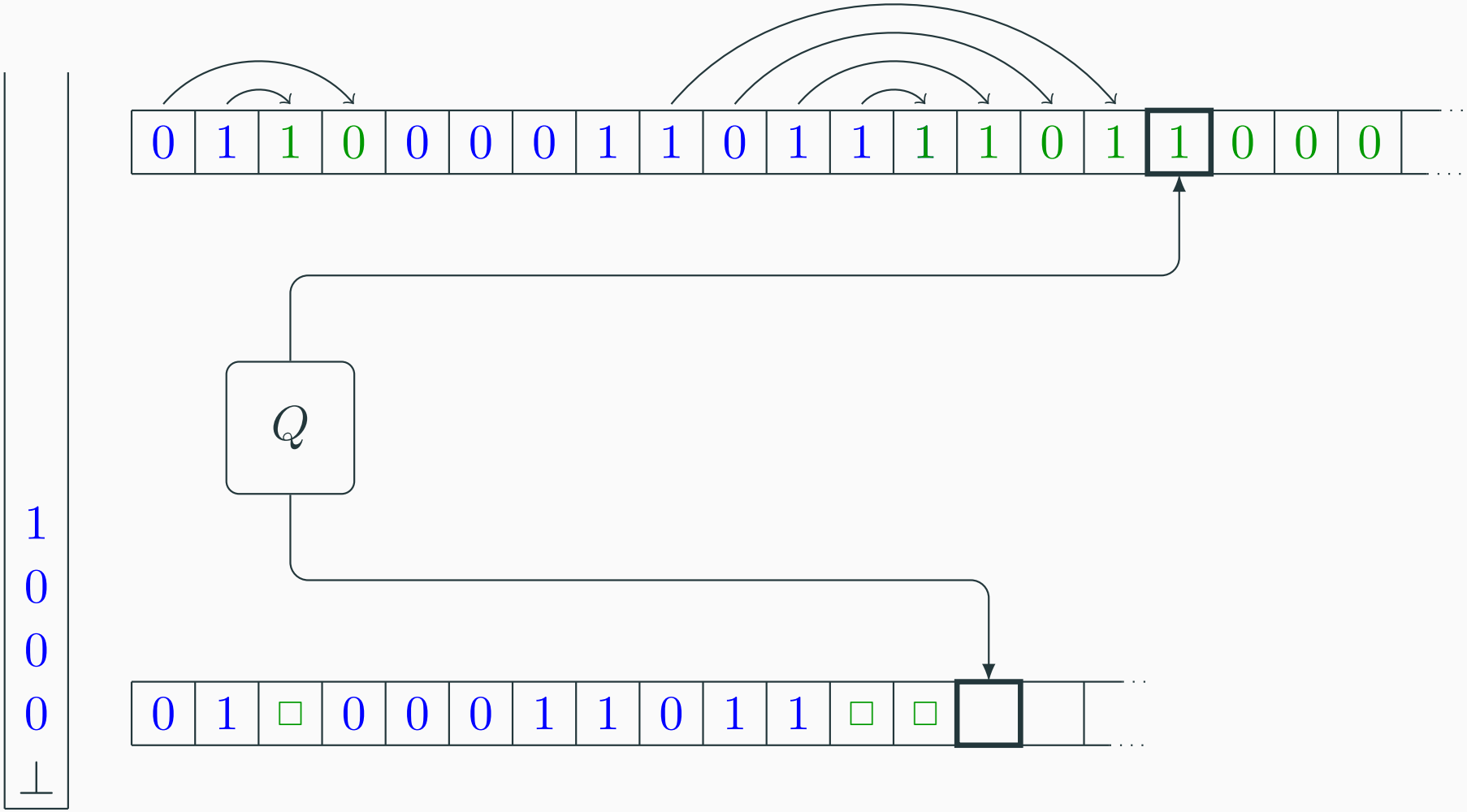

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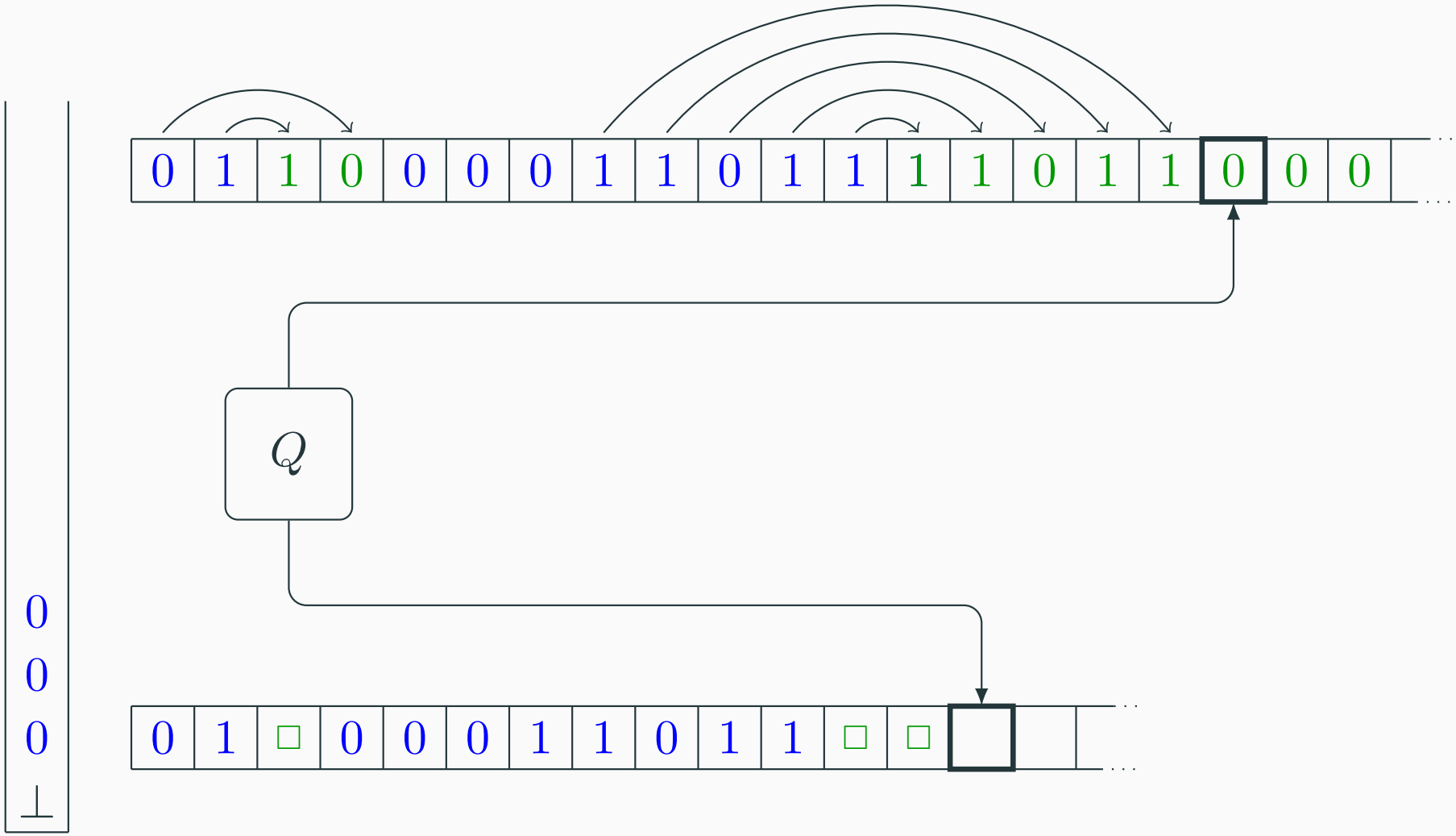

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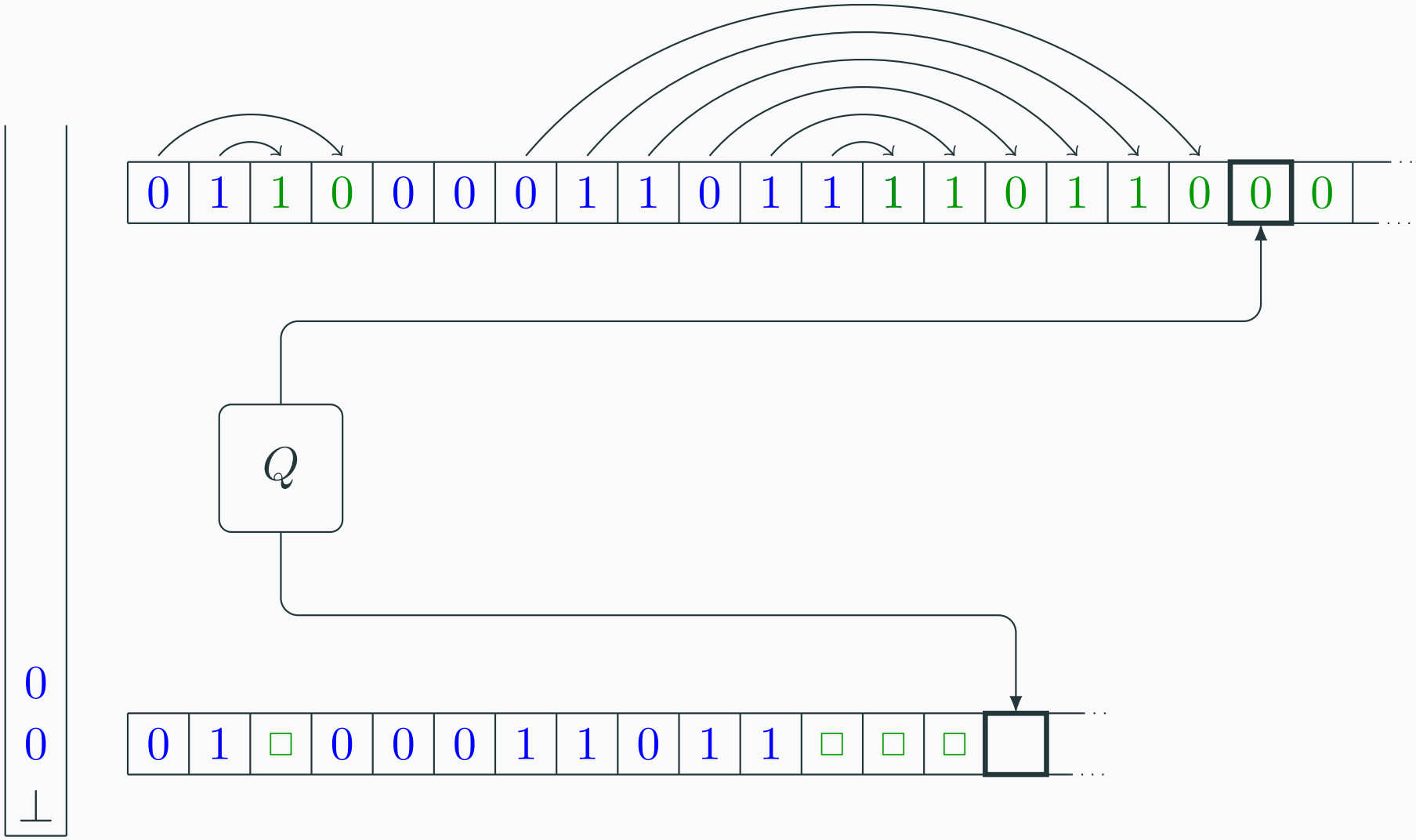

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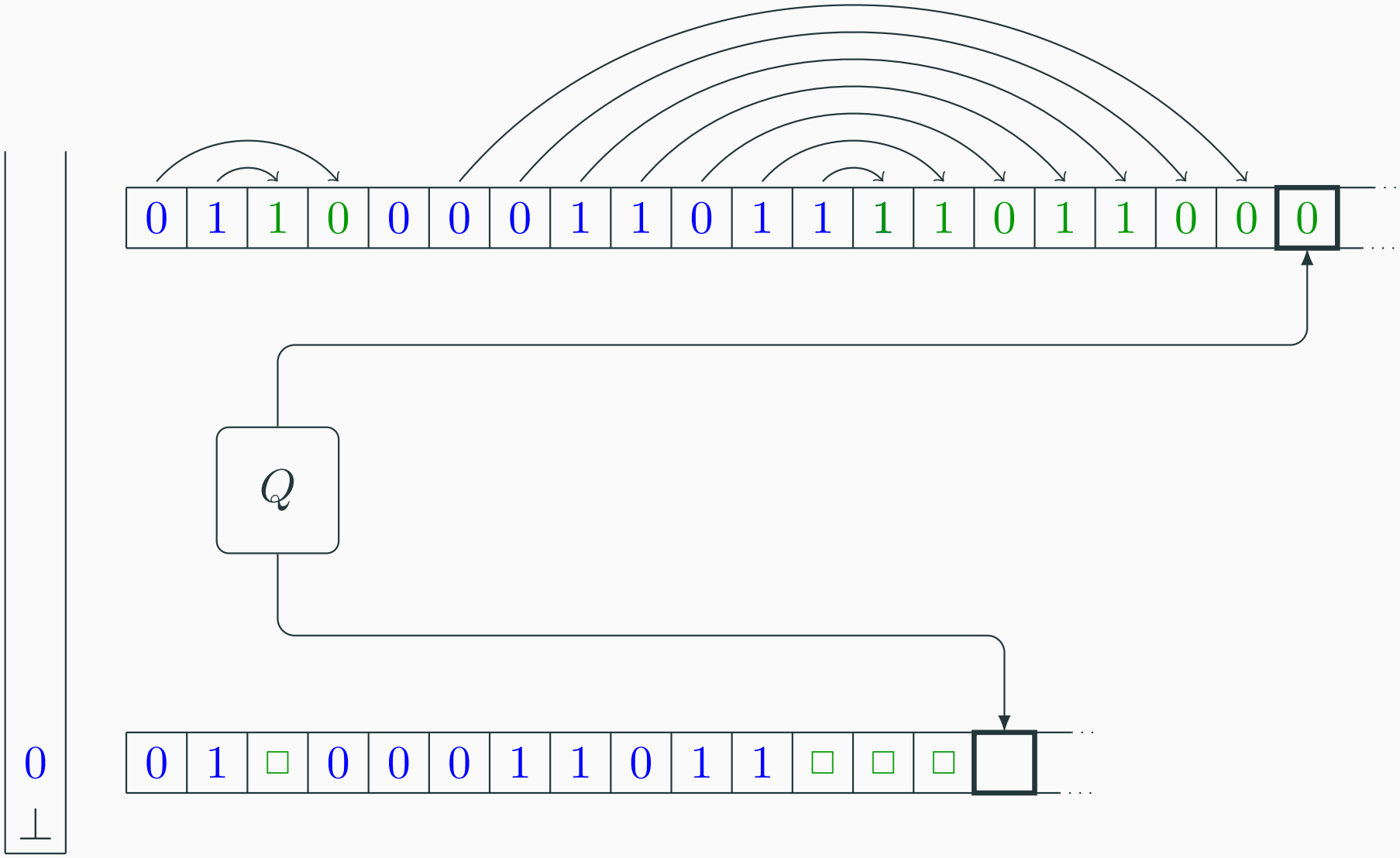

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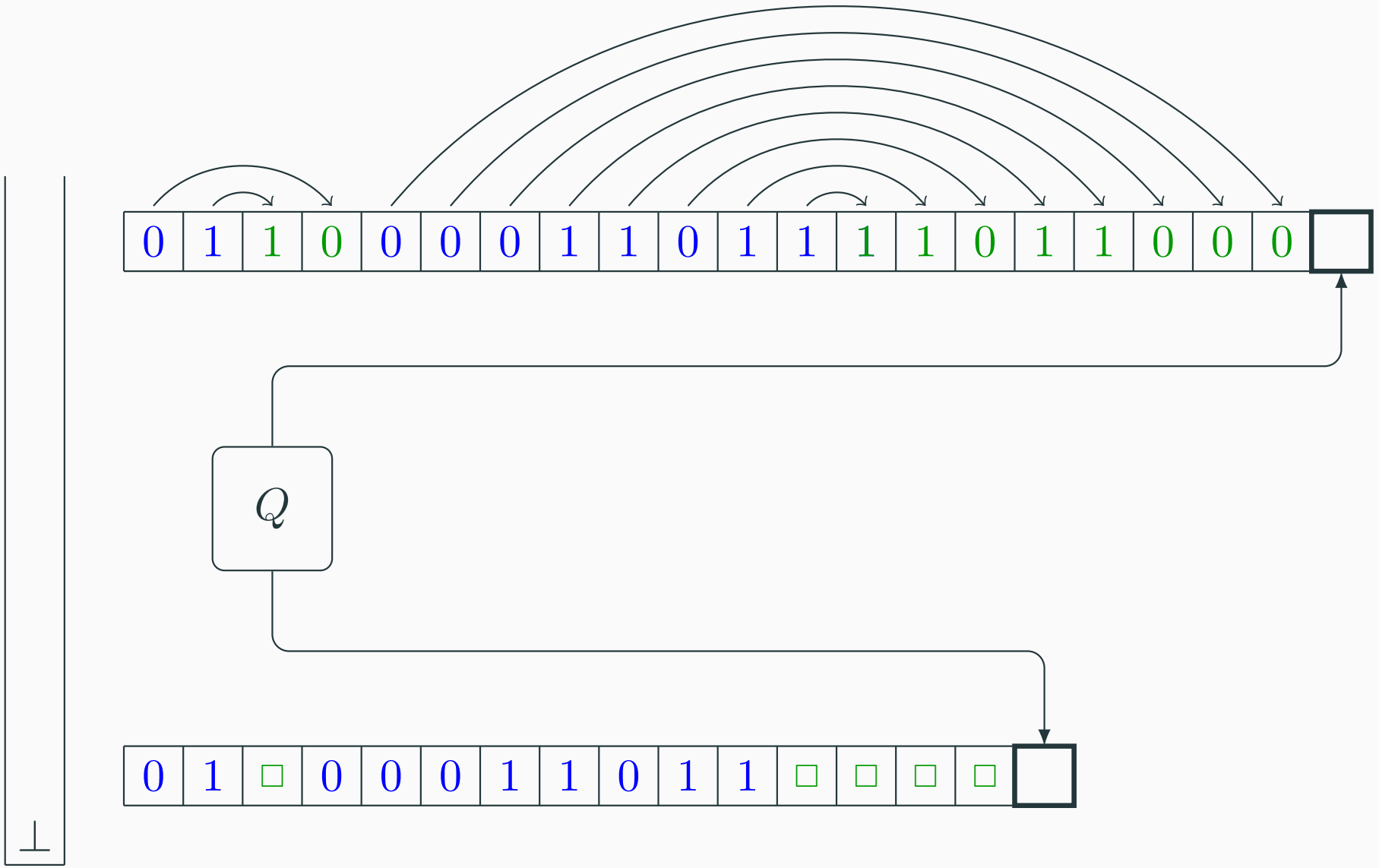


Known result

Theorem

- *The transducer is one-to-one.*
- *The compression ratio is*

$$\frac{3\log(\#A+1)}{4\log(\#A)}.$$

Outline

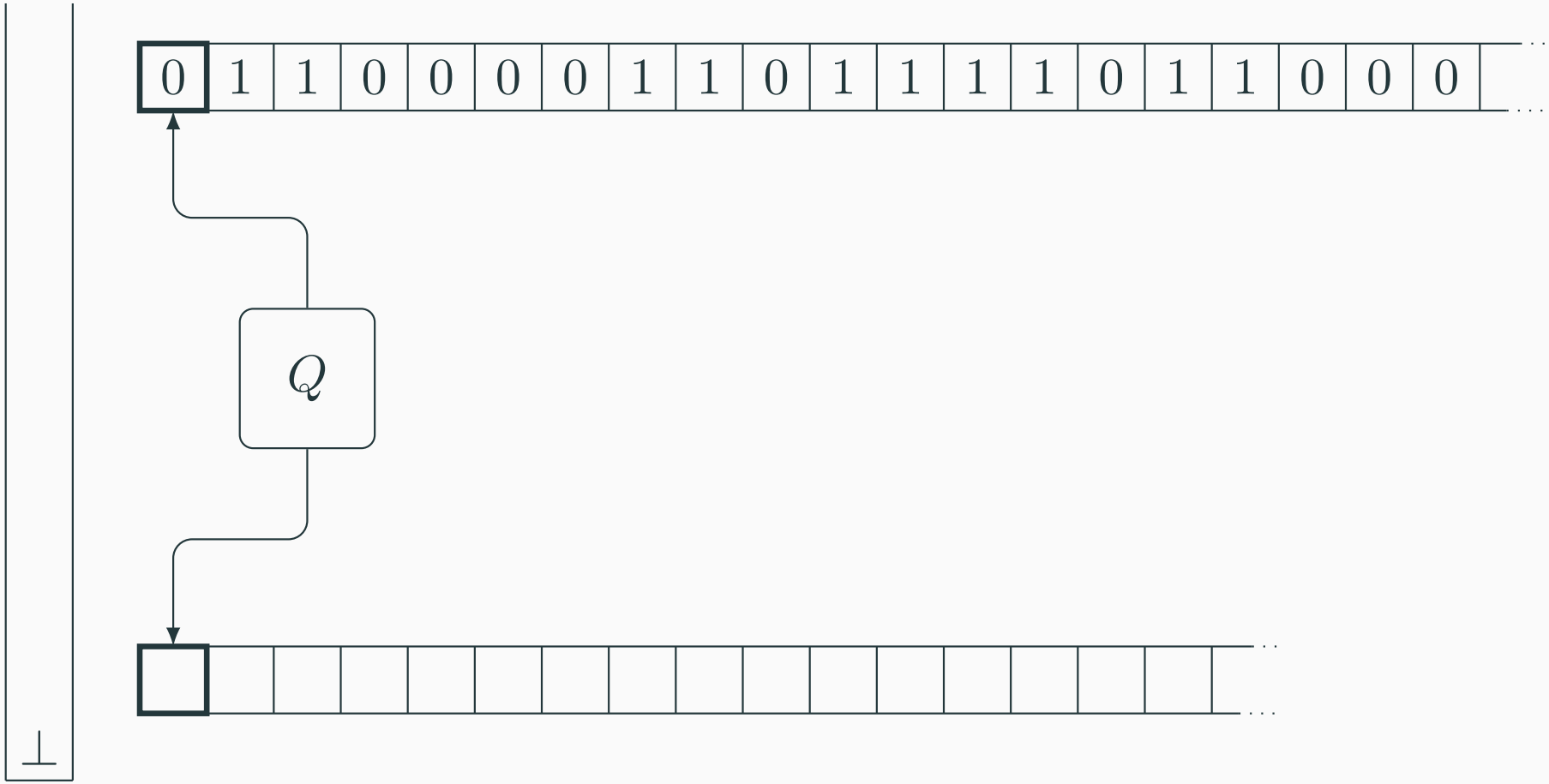
Deterministic pushdown transducers

$$x = w_1\tilde{w}_1 w_2\tilde{w}_2 w_3\tilde{w}_3 \cdots$$

The transducer works as follows.

- If the input symbol is different from stack top symbol, the input symbol is pushed and output,
- If the input symbol and the stack top symbol are equal, the top symbol is popped.
  A symbol $\square$ is output every two popped symbols and a symbol $\triangle$ is output at the end if the number of popped symbols is odd.

Let it run



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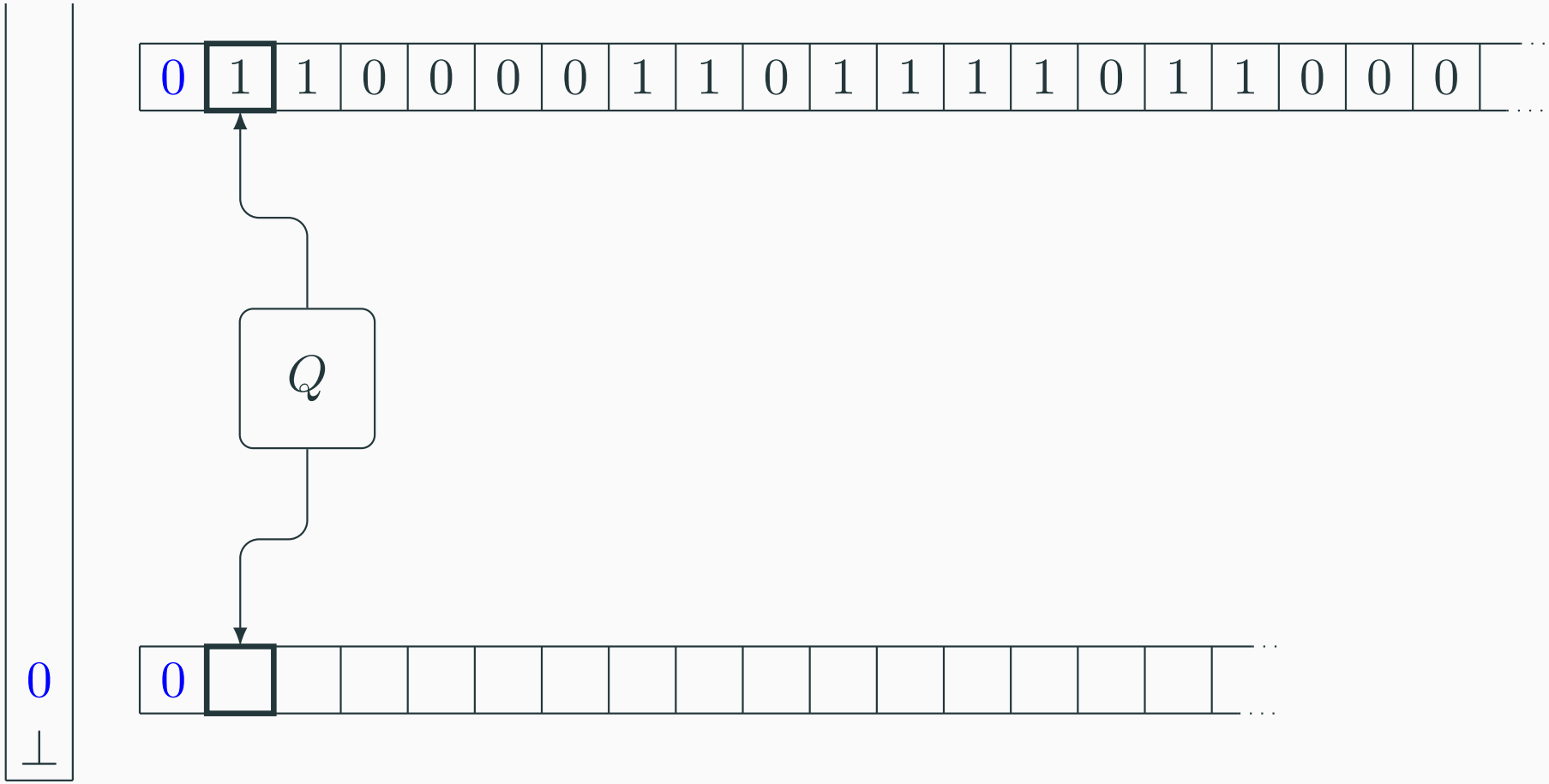

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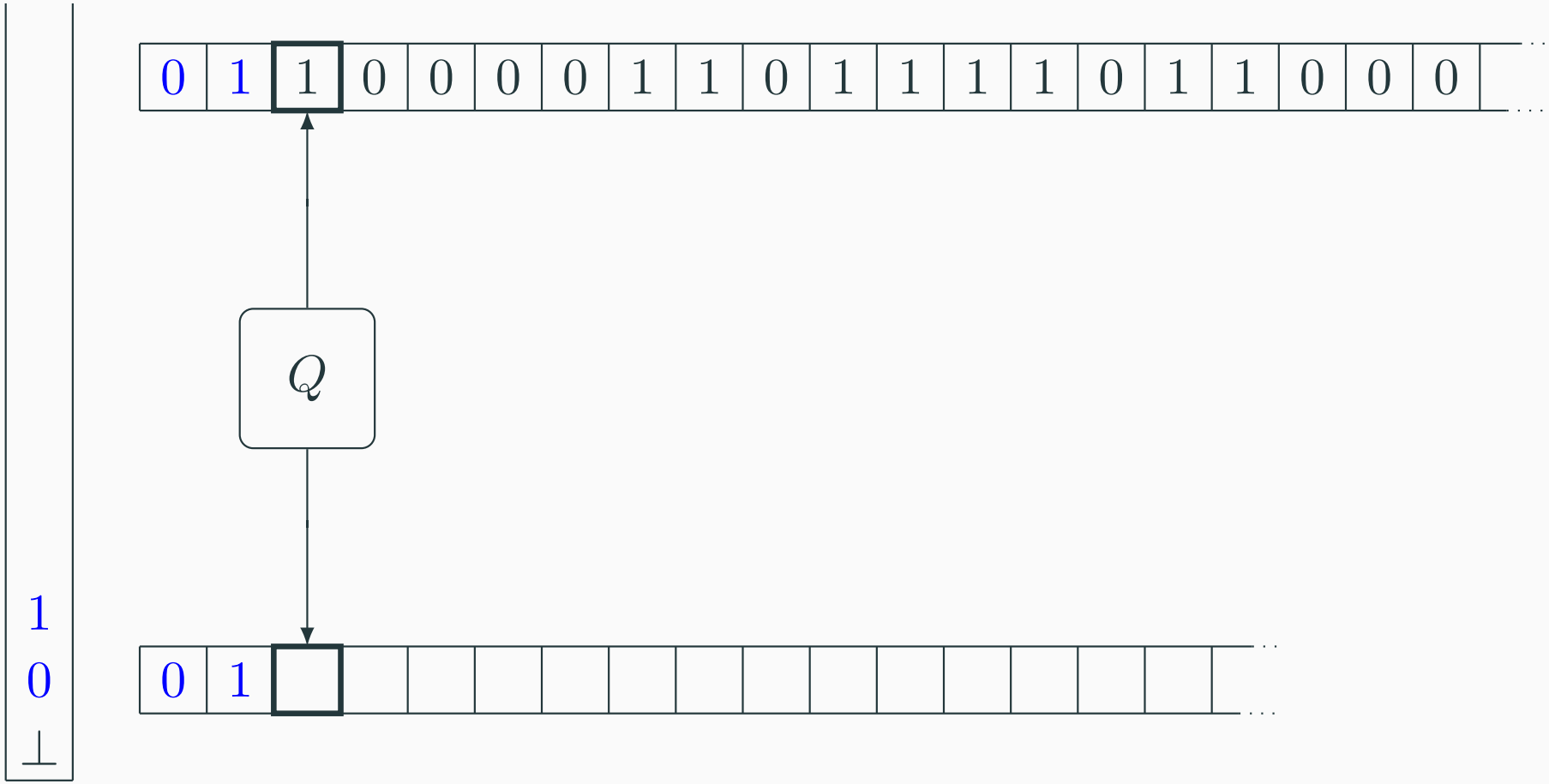

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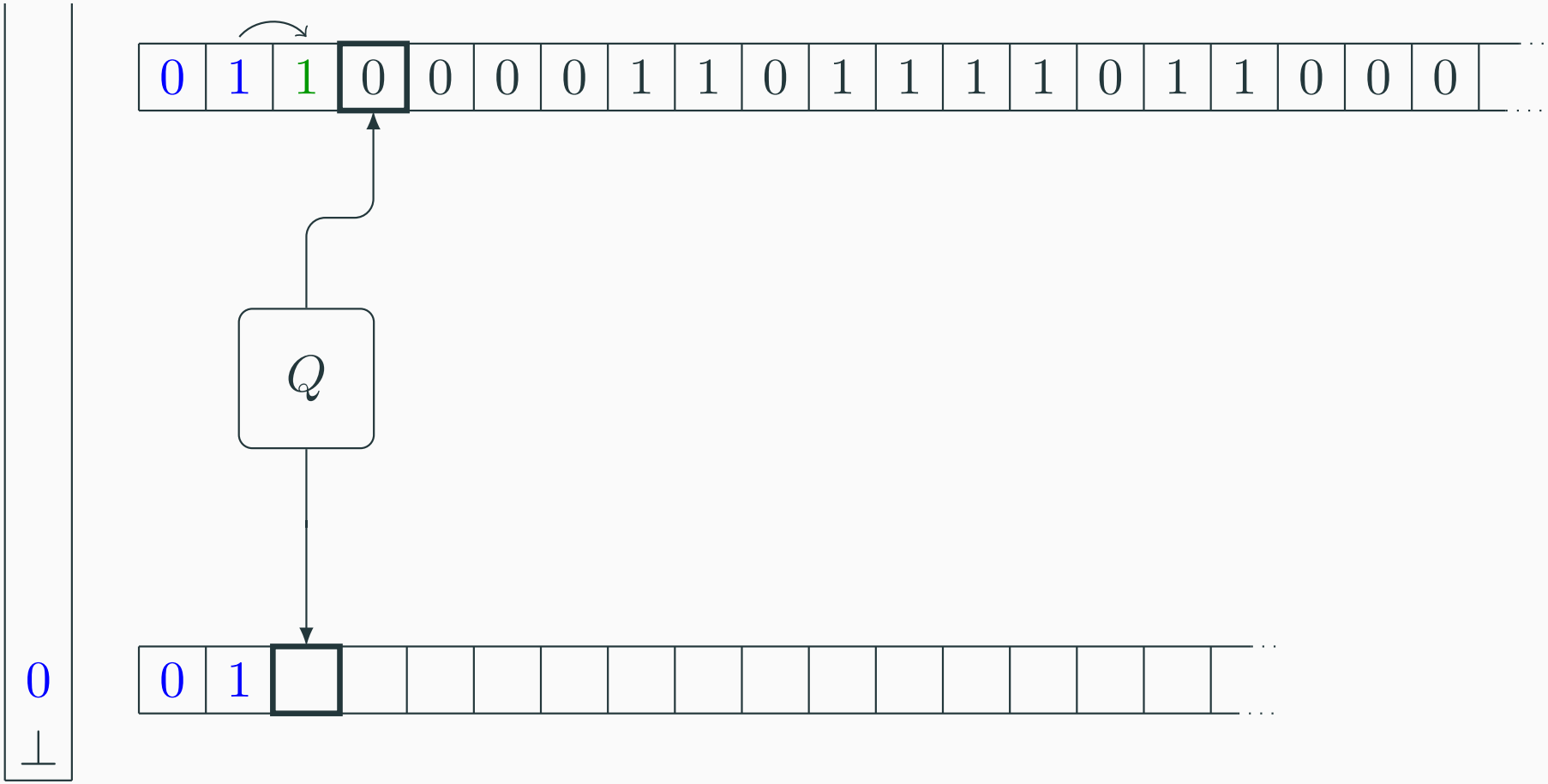

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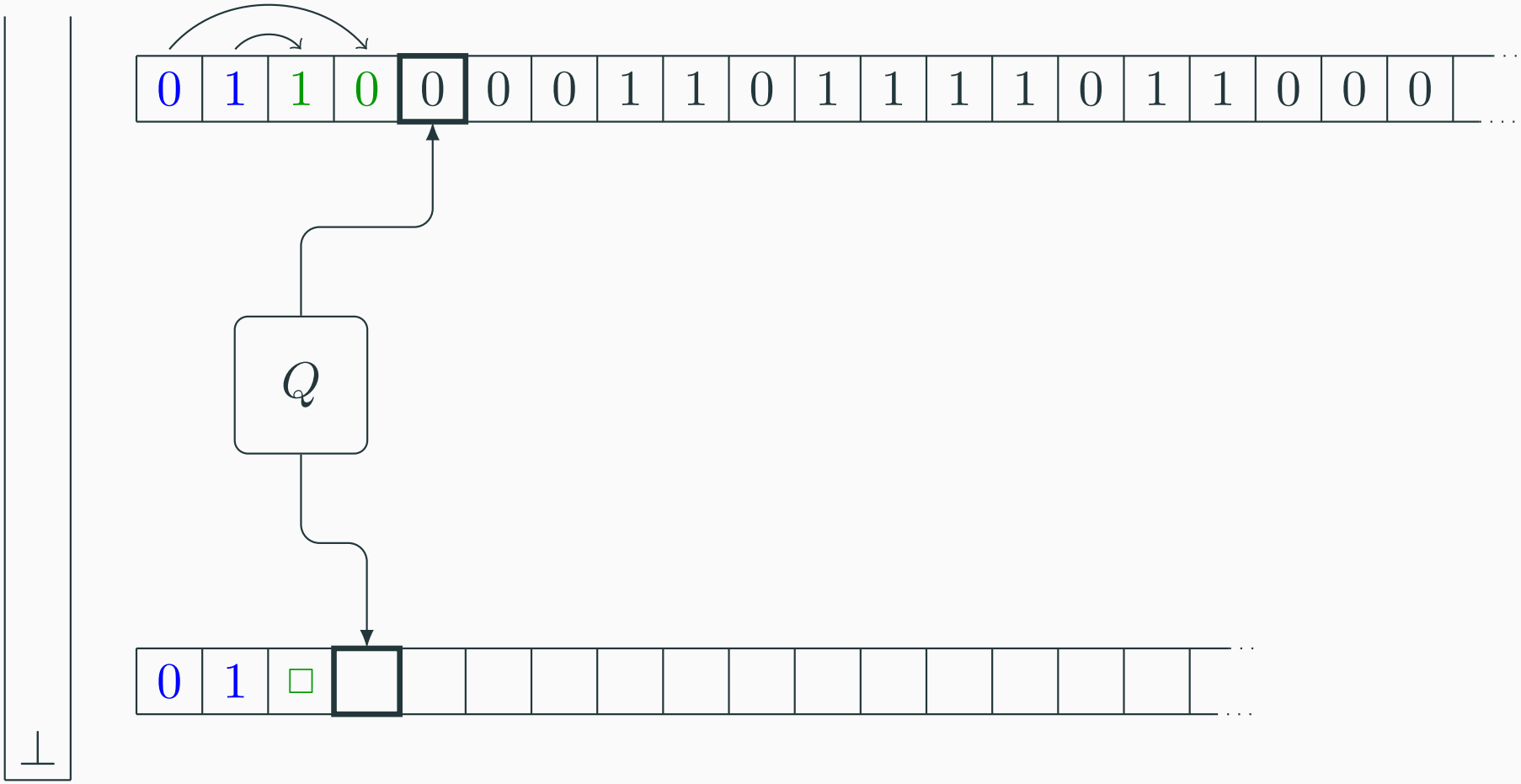

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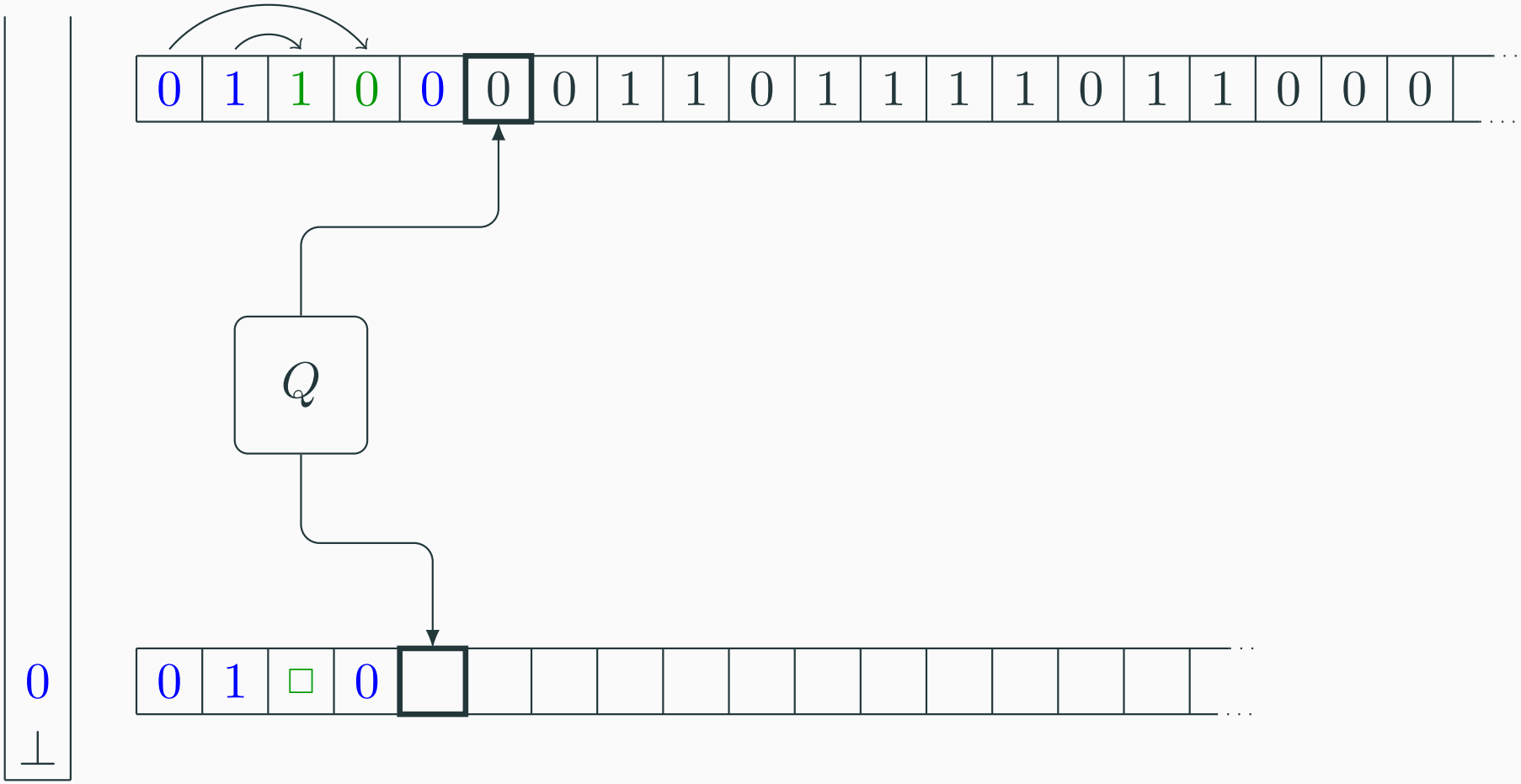

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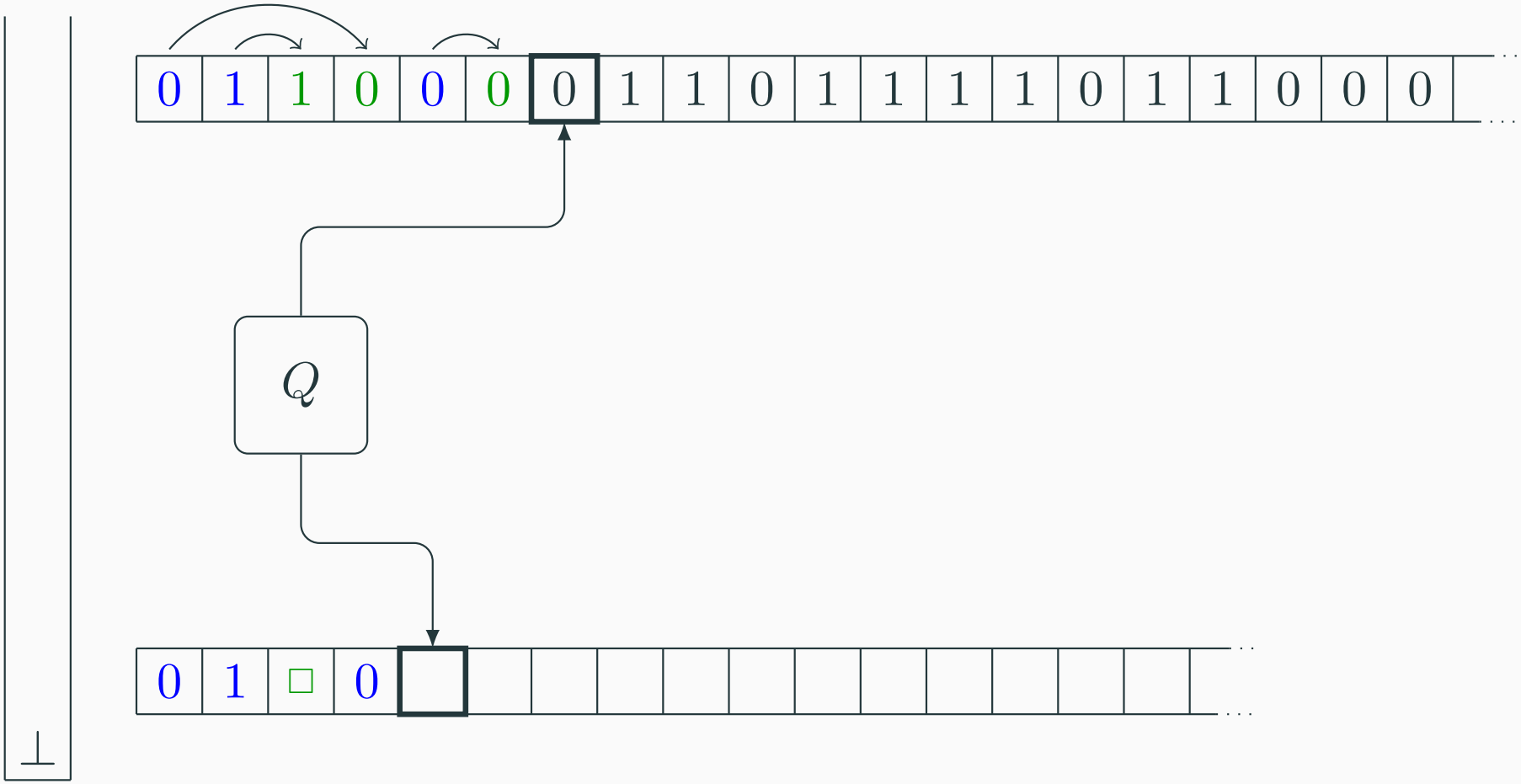

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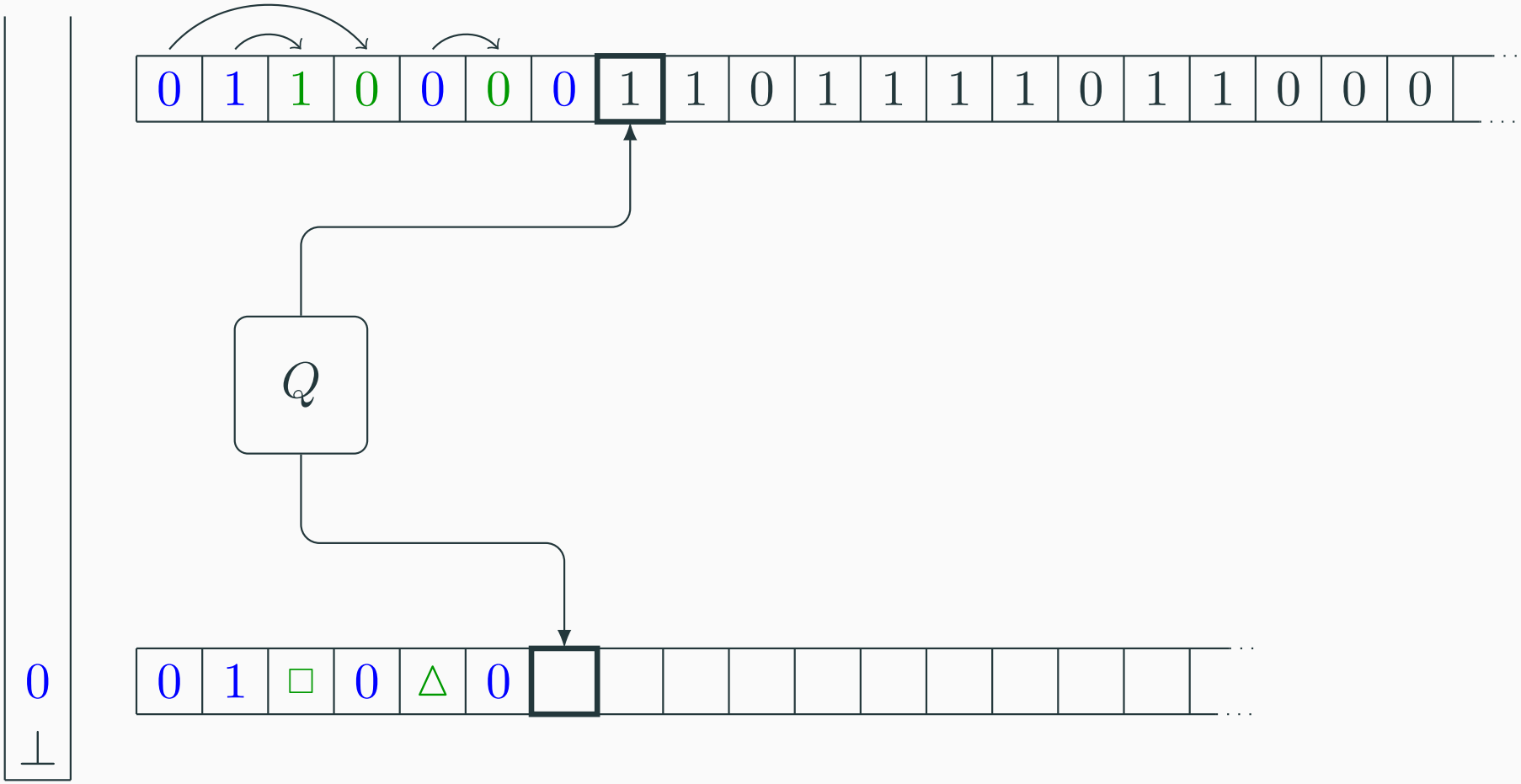

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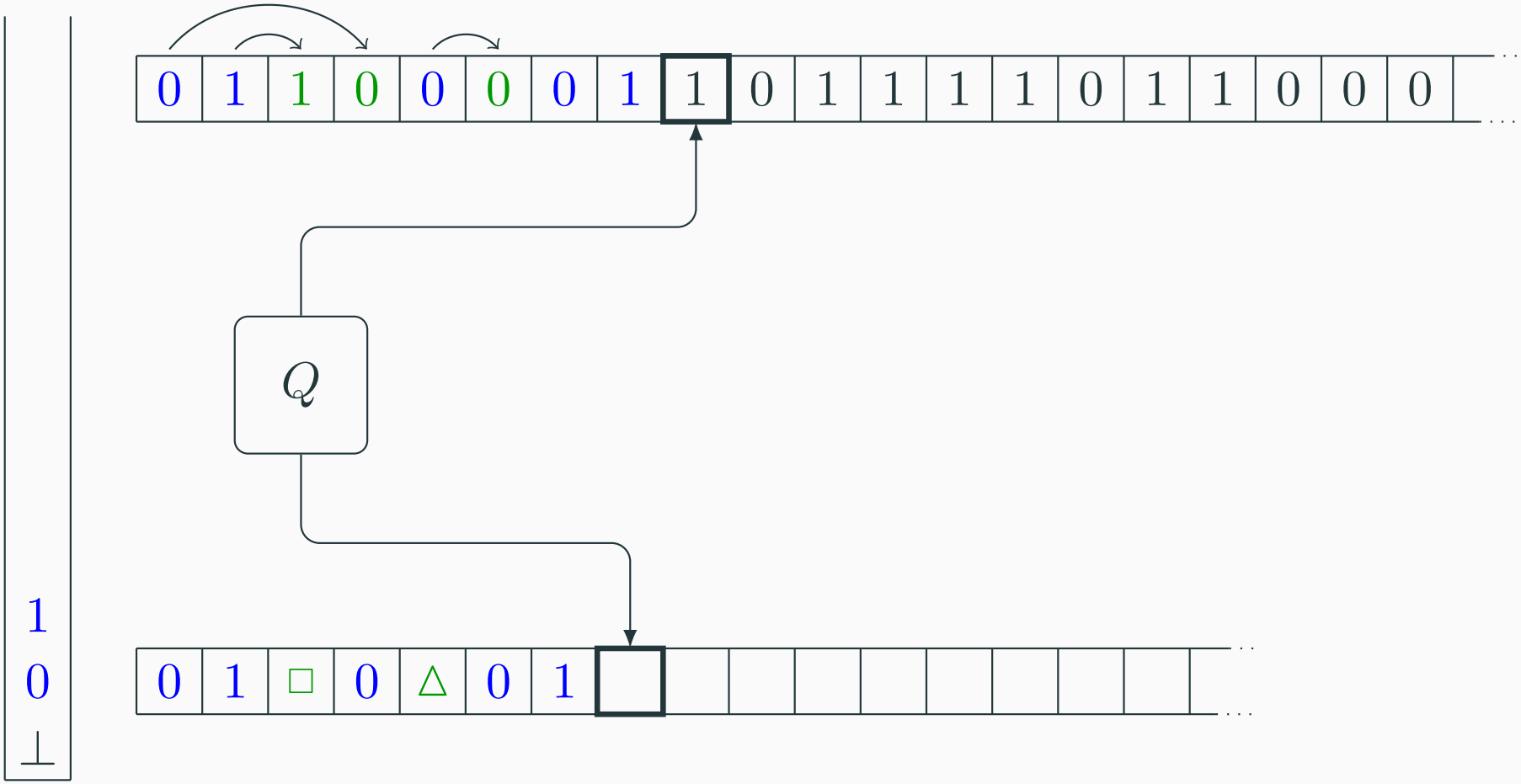

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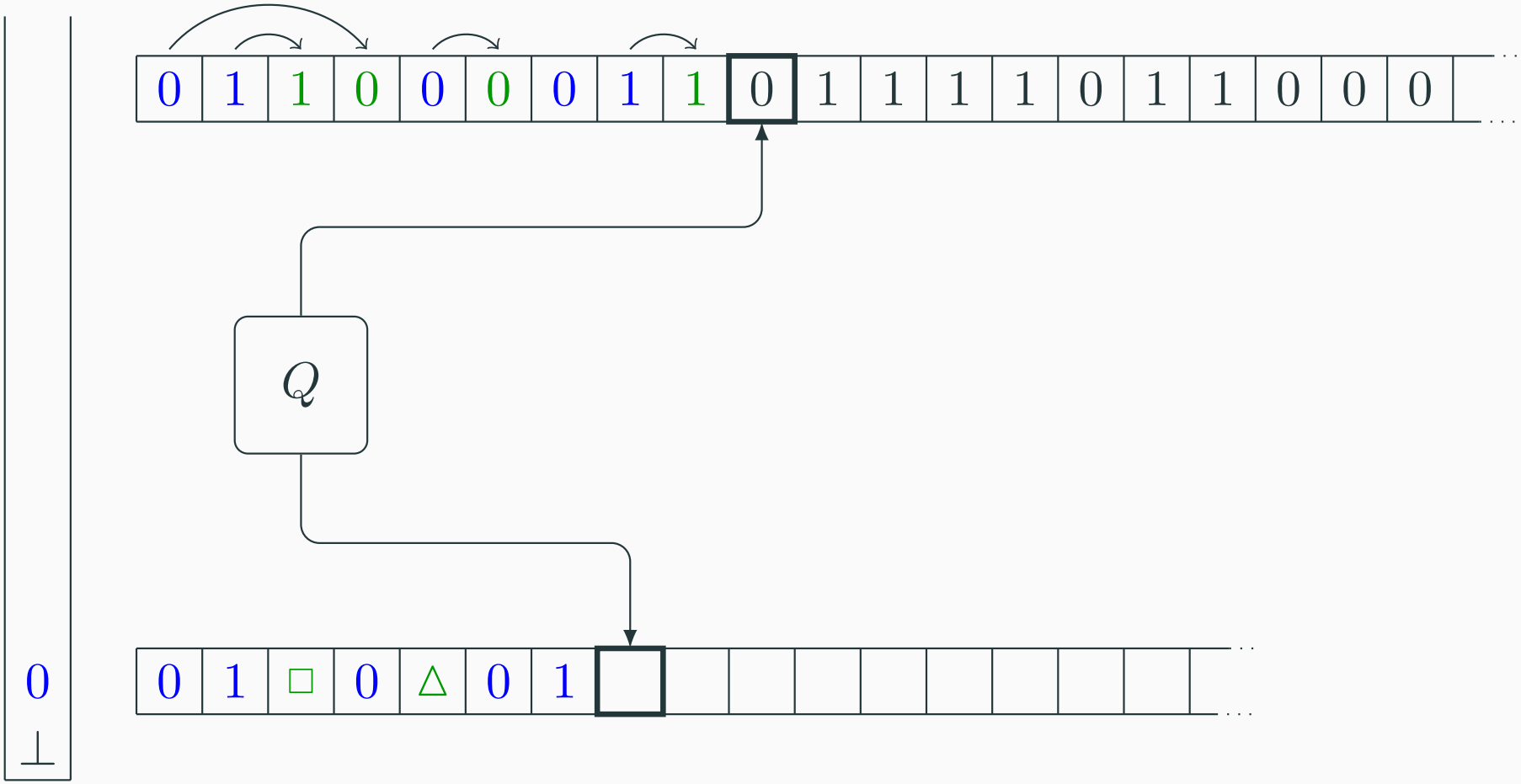

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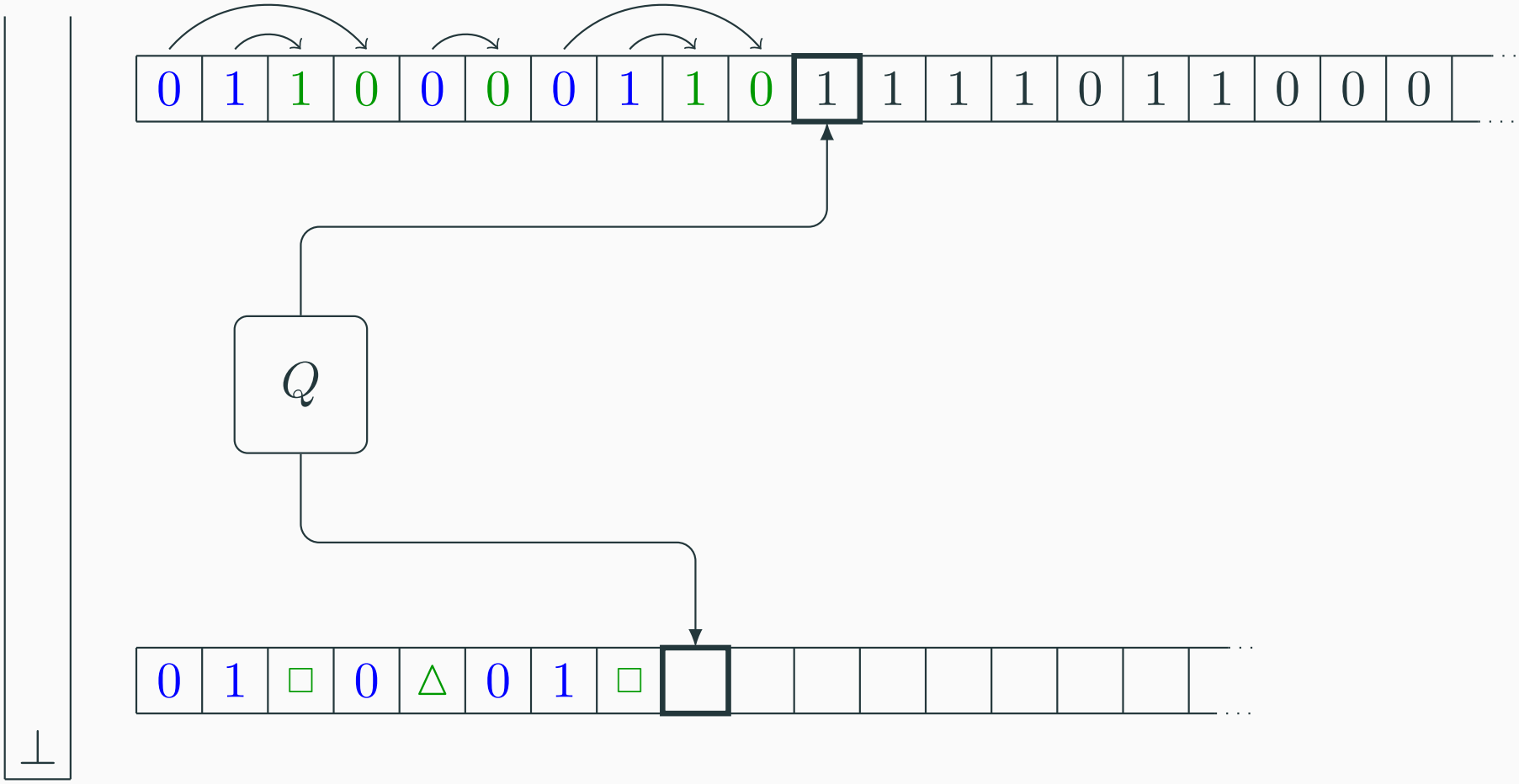

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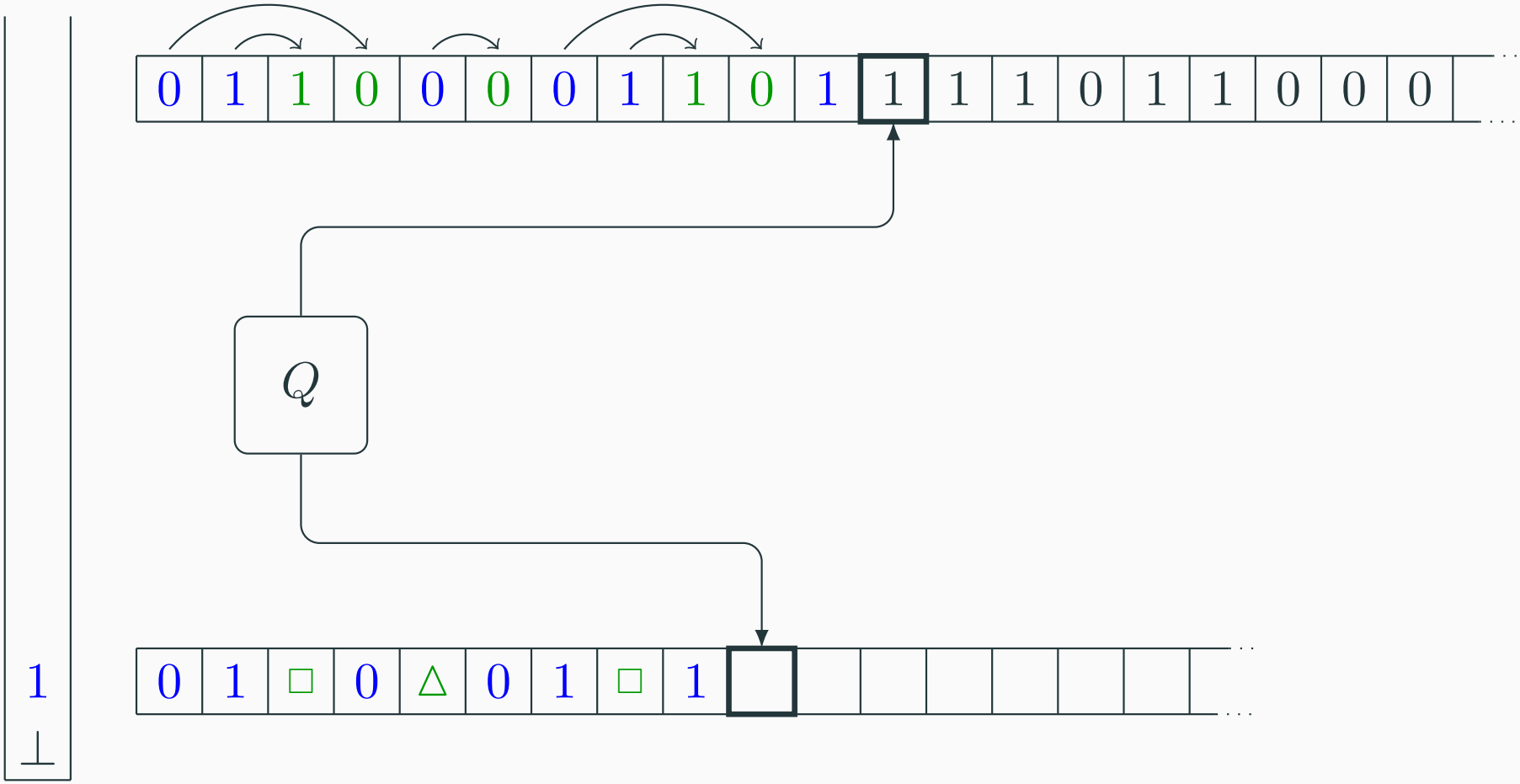

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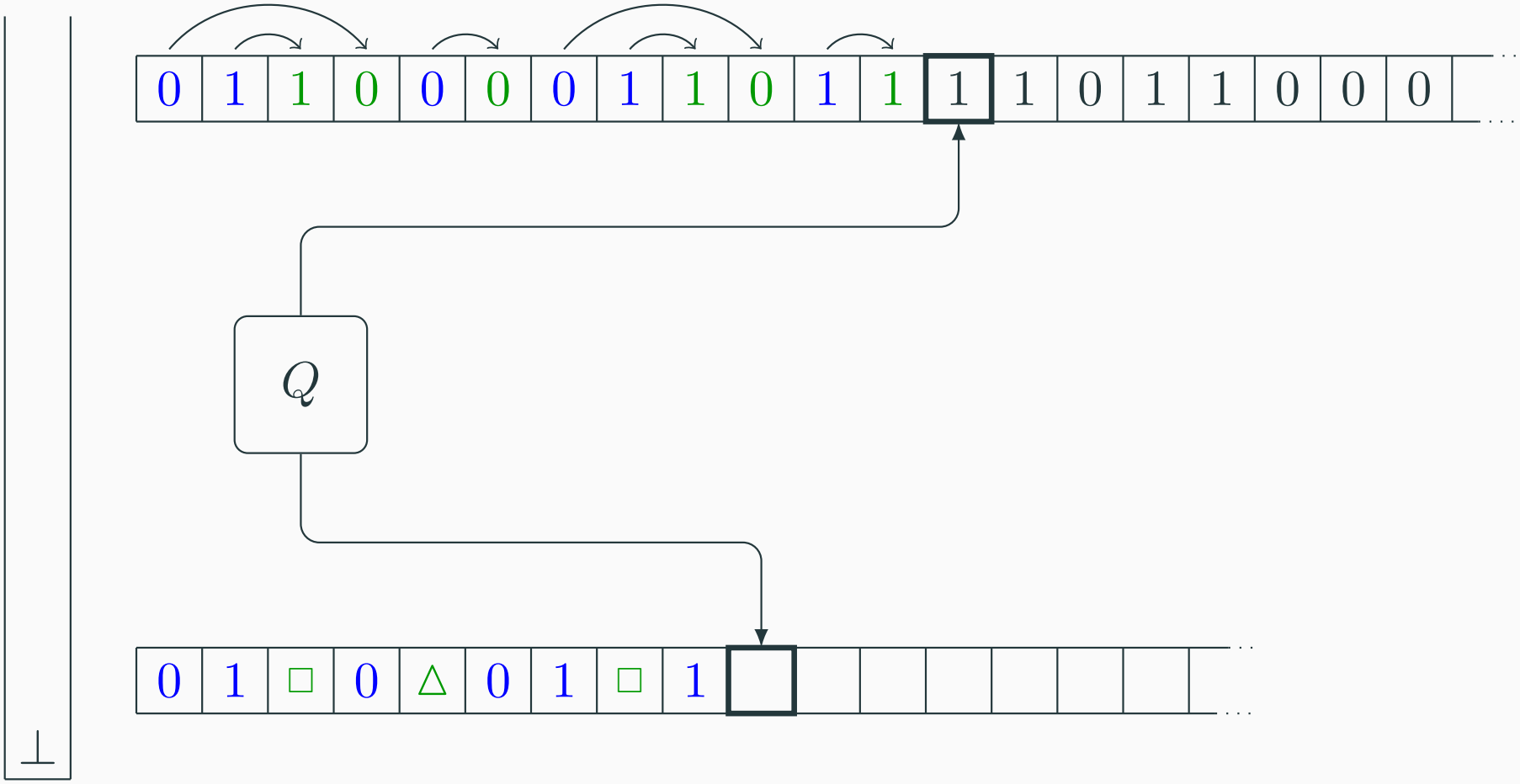

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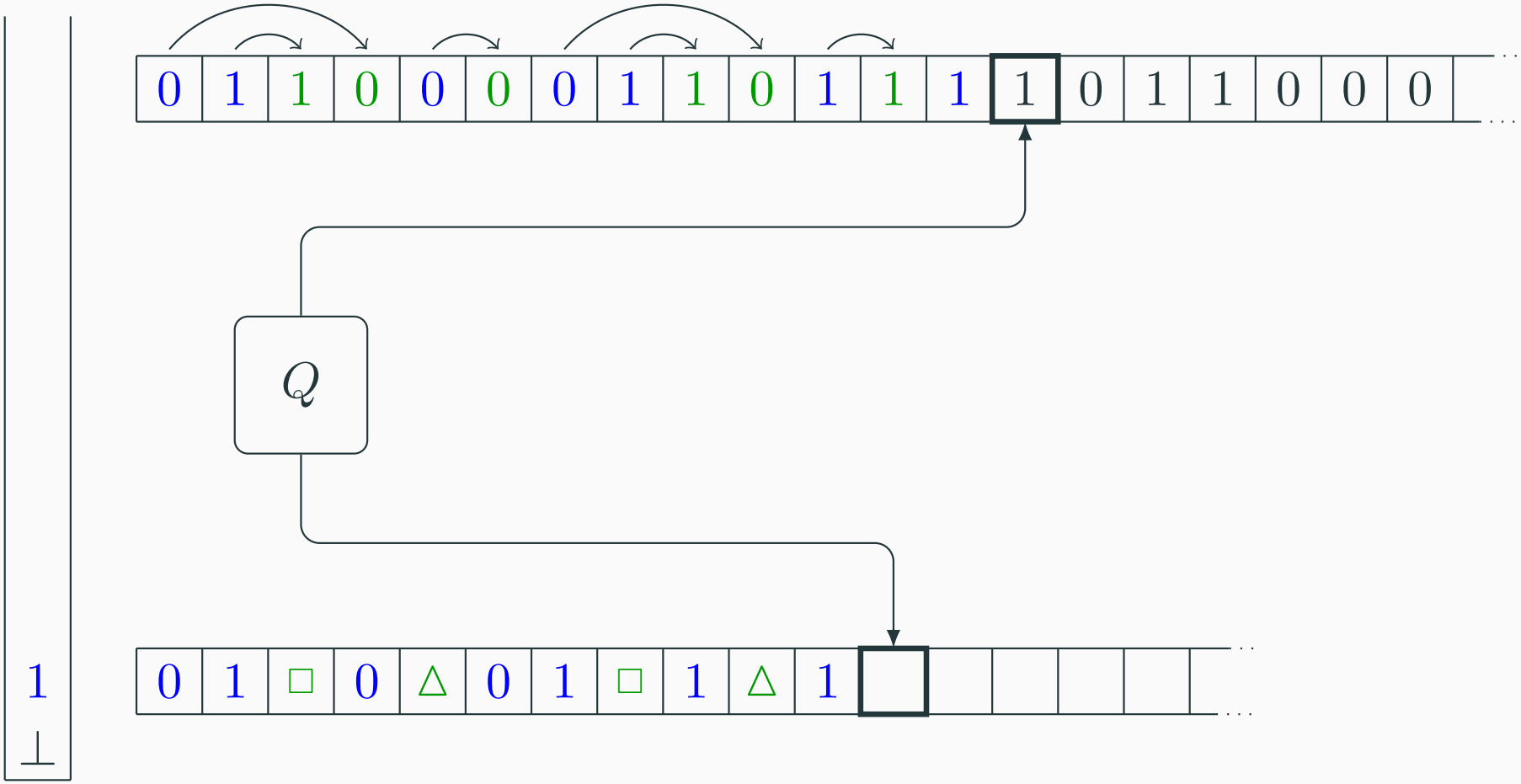

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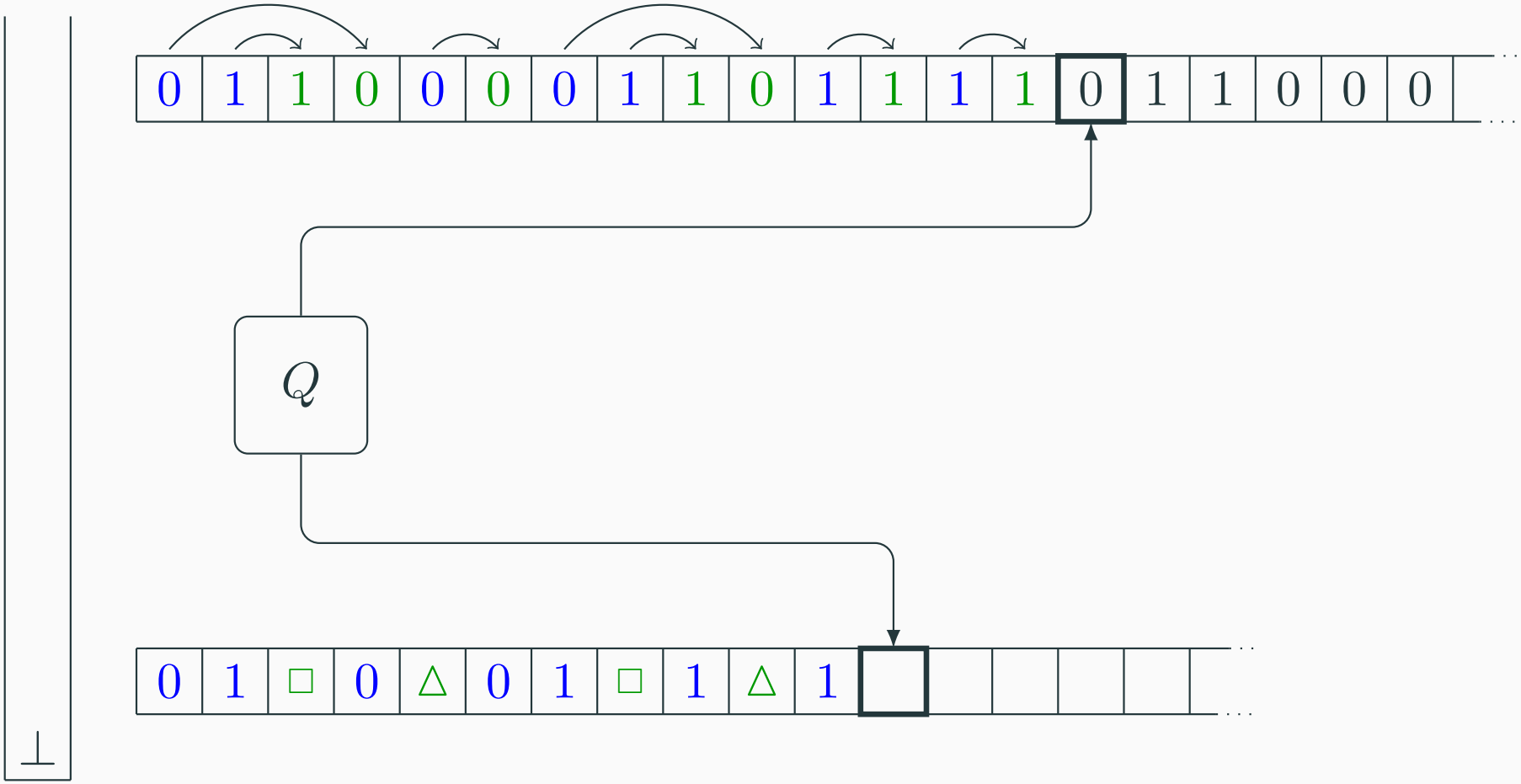

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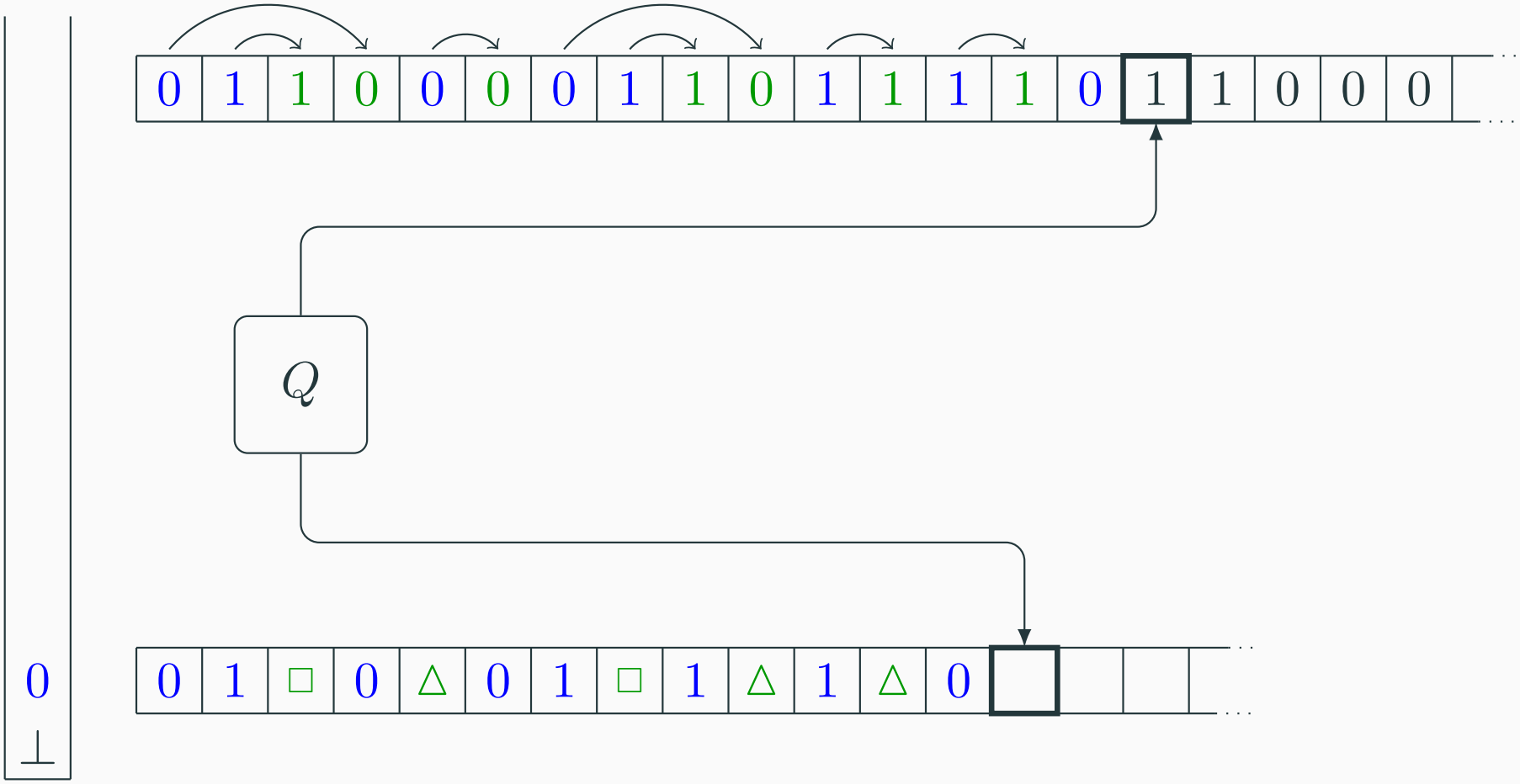

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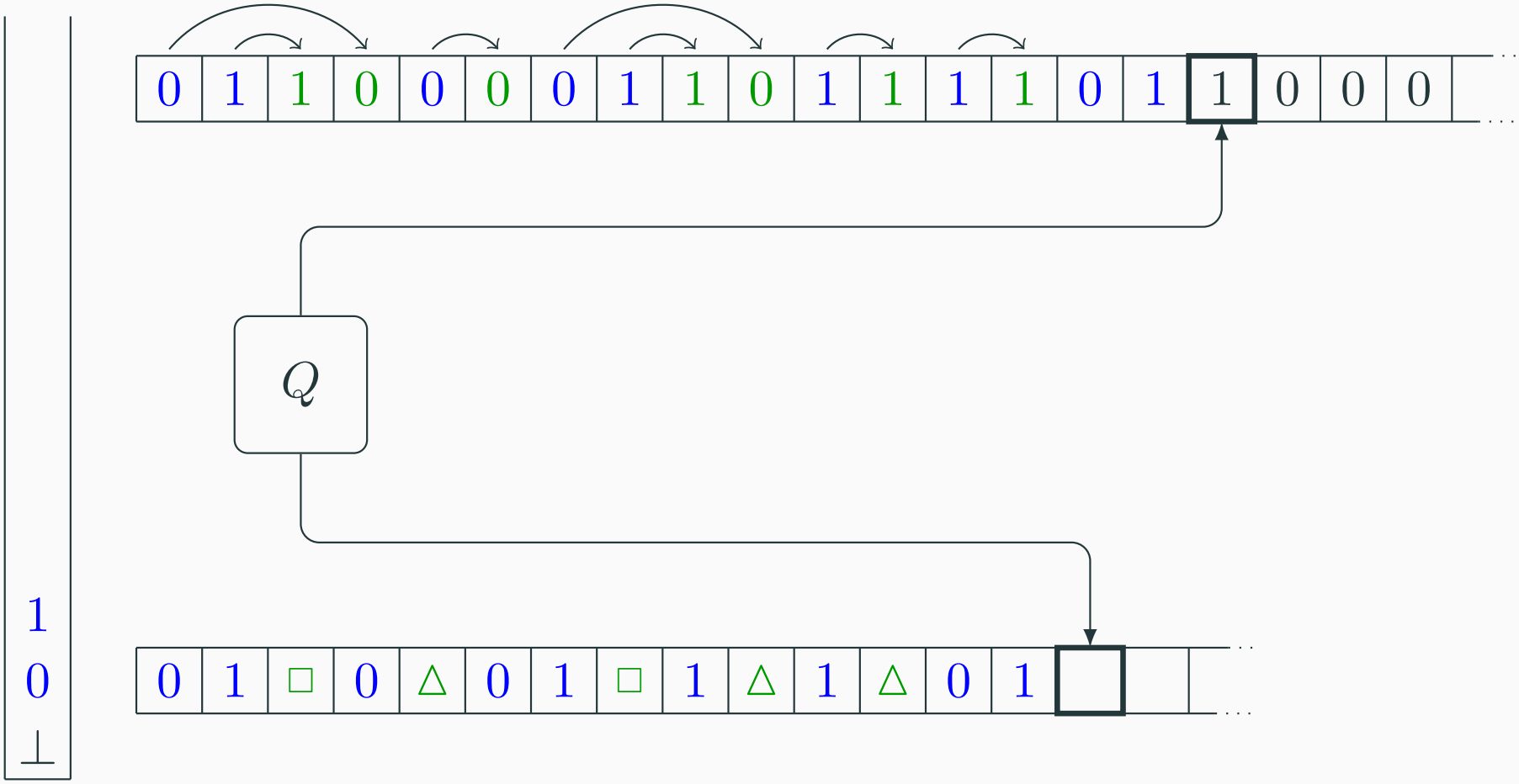

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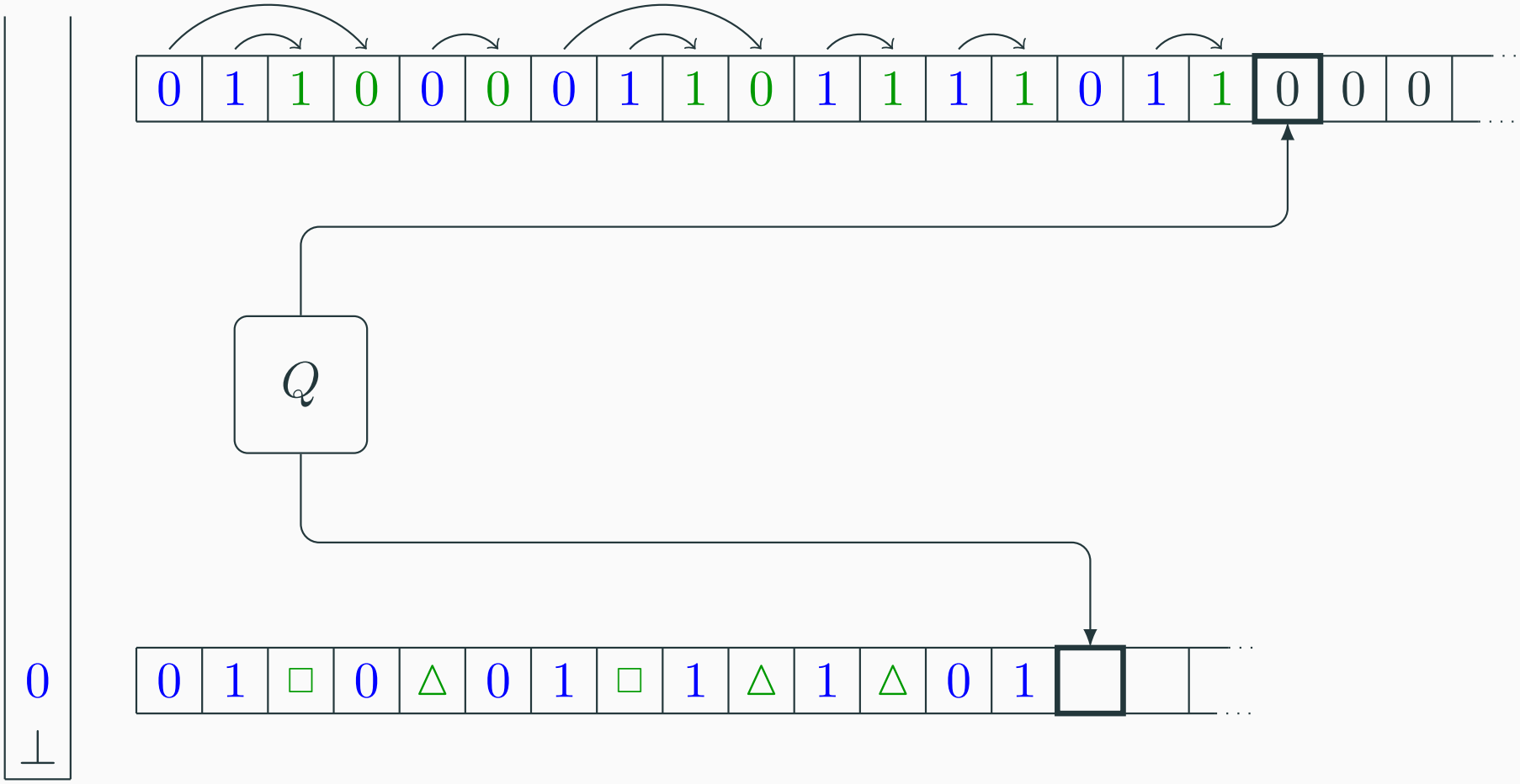

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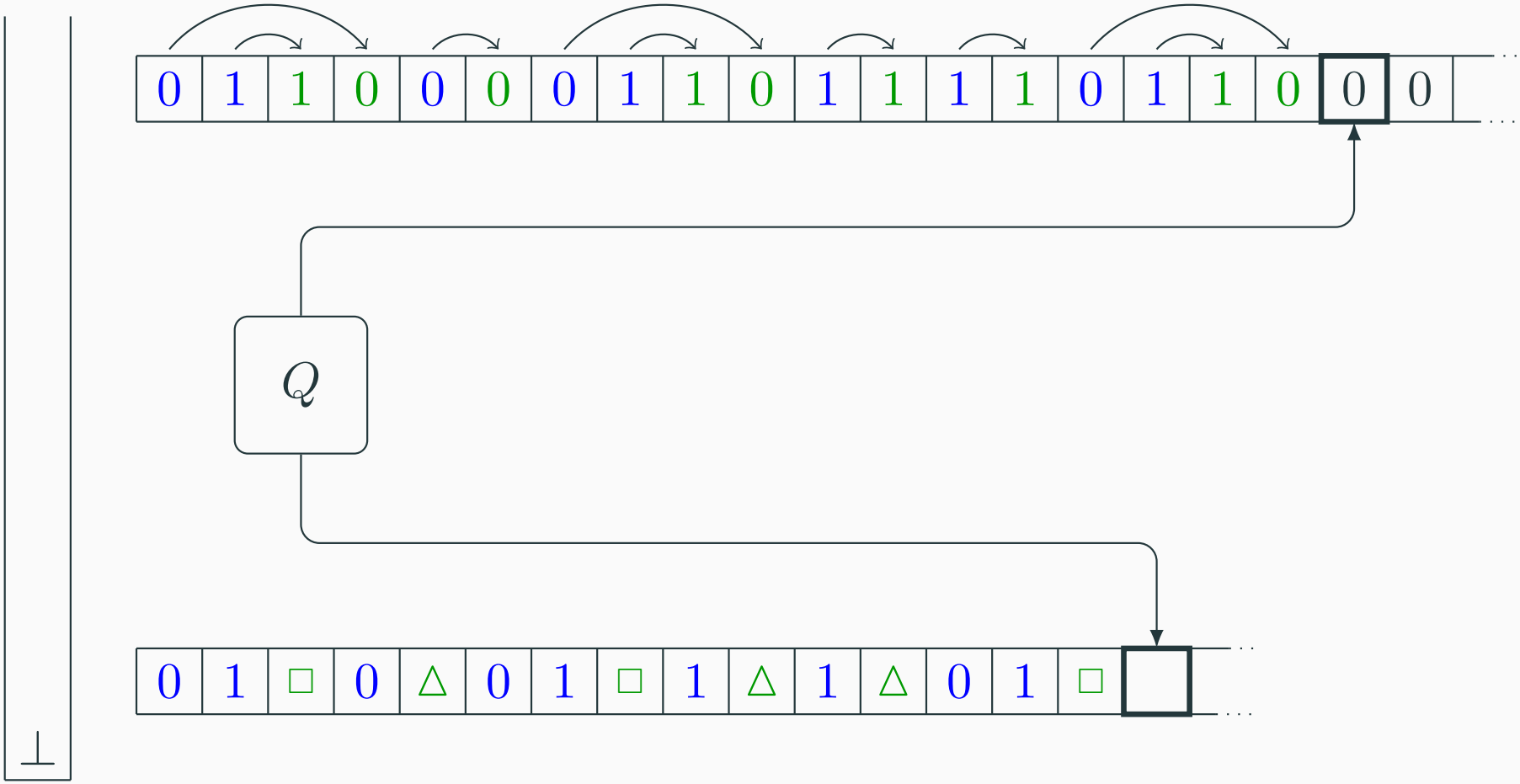

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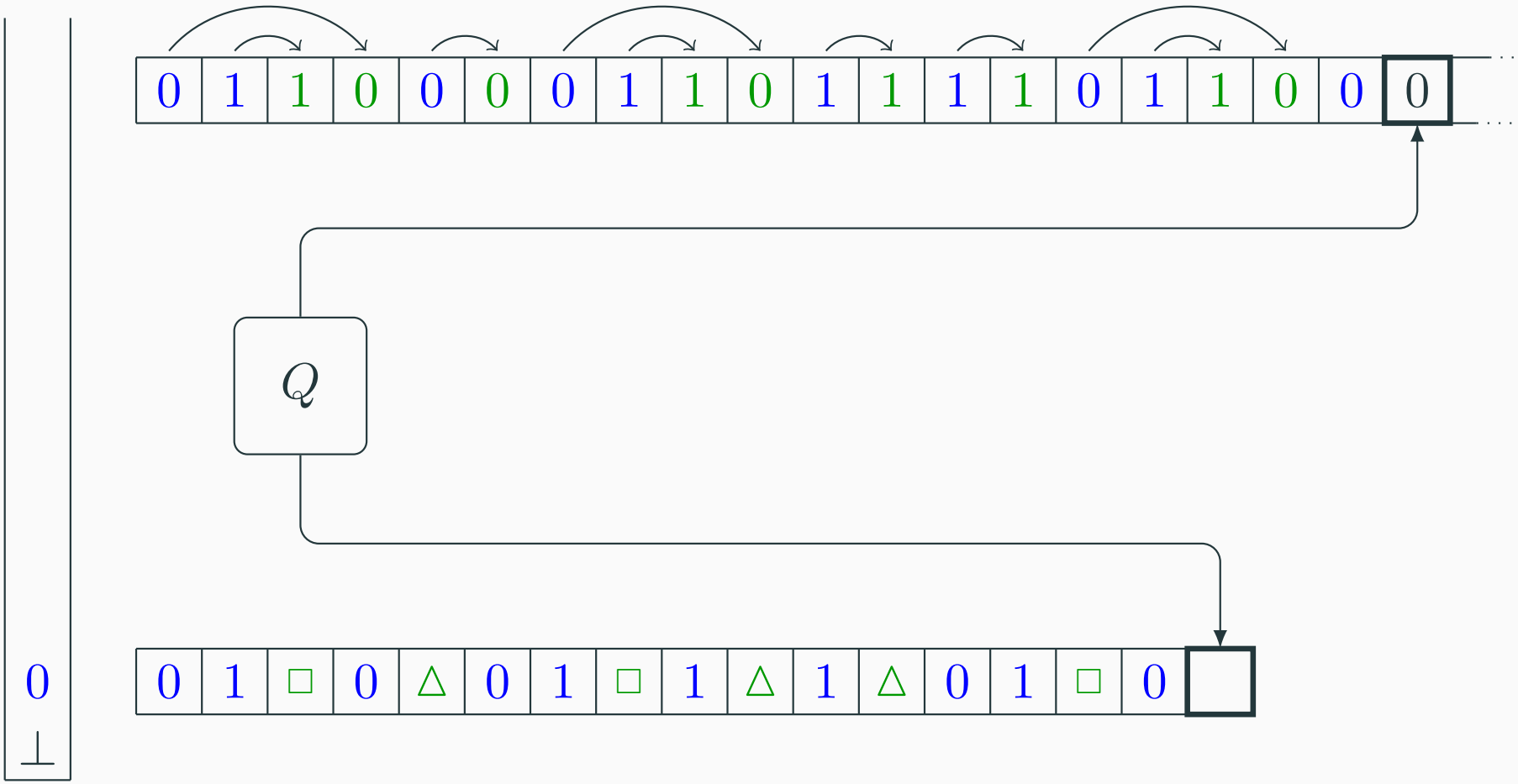

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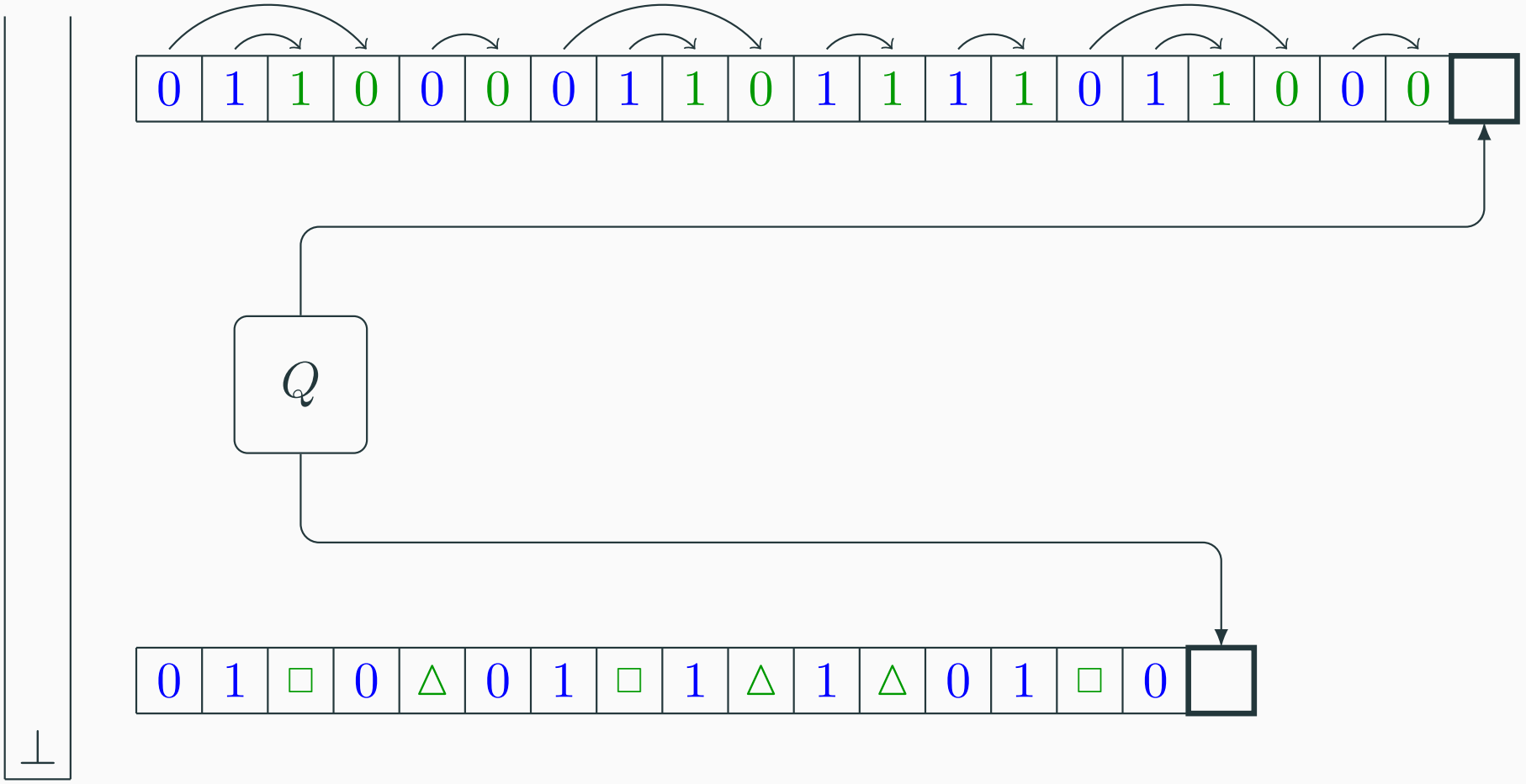


Result

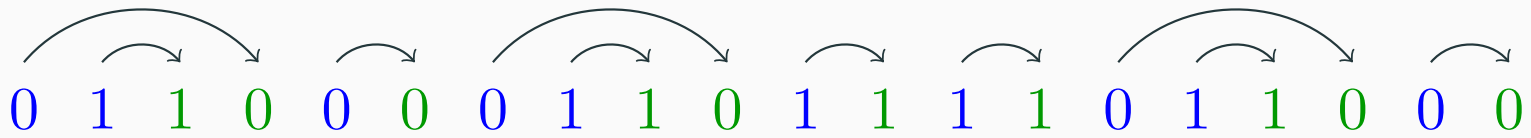
Theorem

- *The transducer is one-to-one.*
- *The compression ratio is less than*

$$\frac{19\log(\#A+2)}{20\log(\#A)}$$

for $\#A$ large enough.

Sketch of proof



- Each pair of consecutive pops saves one symbol.
- Each long edge implies a pair of consecutive pops.
- Each block of odd length implies a long edge.
- If $\#A$ is large, there are many blocks of length 1 in $w_n\tilde{w}_n$.

Open questions

- Which sequences are compressed by this transducer ?
- Does the compression ratio converge to $3/4$ when the alphabet size k goes to infinity ?
- ...

Merci