Separation and Concatenations

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LaBRI, Bordeaux University

February 9, 2017

Certified “100% transducer-free”

Investigating Logics over Words

First-Order Logic over Words (FO(<))

$$a\ b\ b\ b\ c\ a\ a\ a\ c\ a\ \in A^*$$
$$0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9$$

- A word is a sequence of labeled positions.
- Positions can be quantified.

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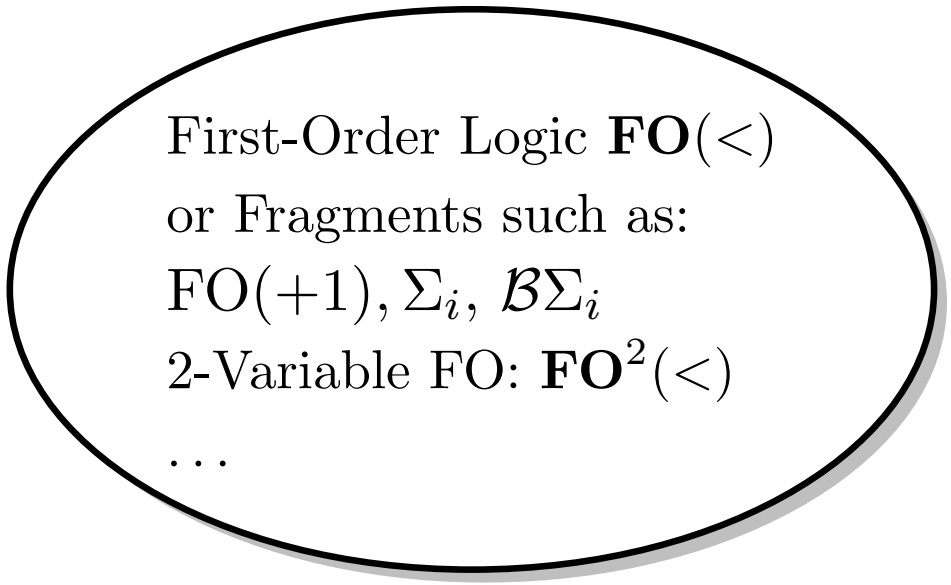
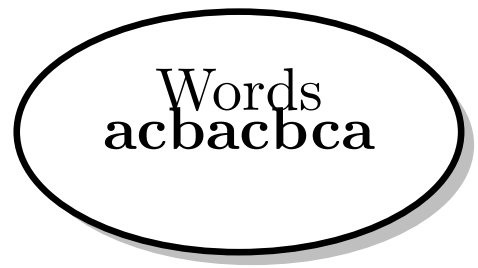
$$\forall x\ (a(x) \Rightarrow \exists y\ (b(y) \wedge (y < x)))$$
"for any a in the word, there is a b to its left"

Each sentence defines a language
$\Rightarrow$ FO(<) defines a **class of languages**.

Main Objective

Descriptive Formalisms





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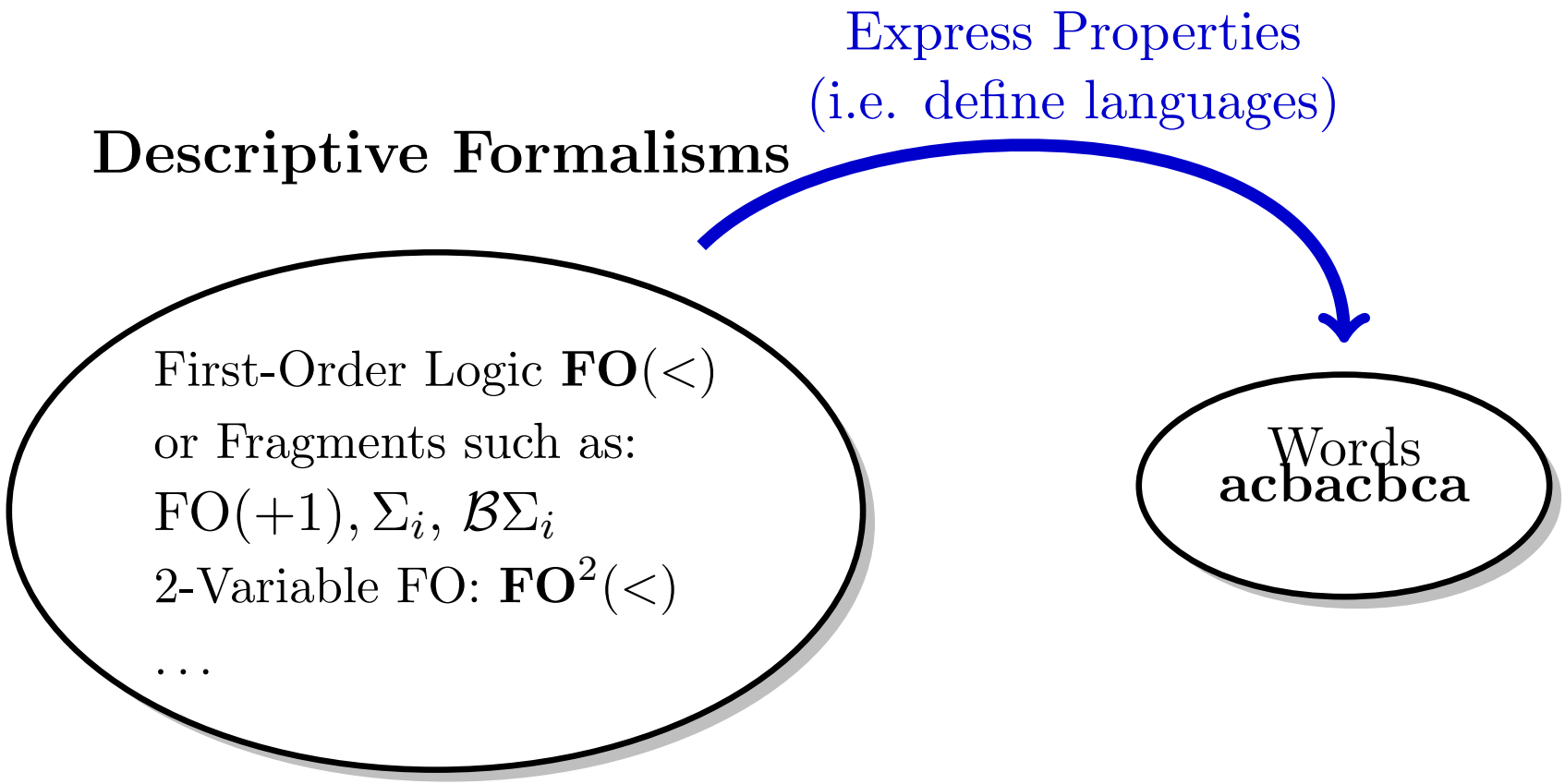

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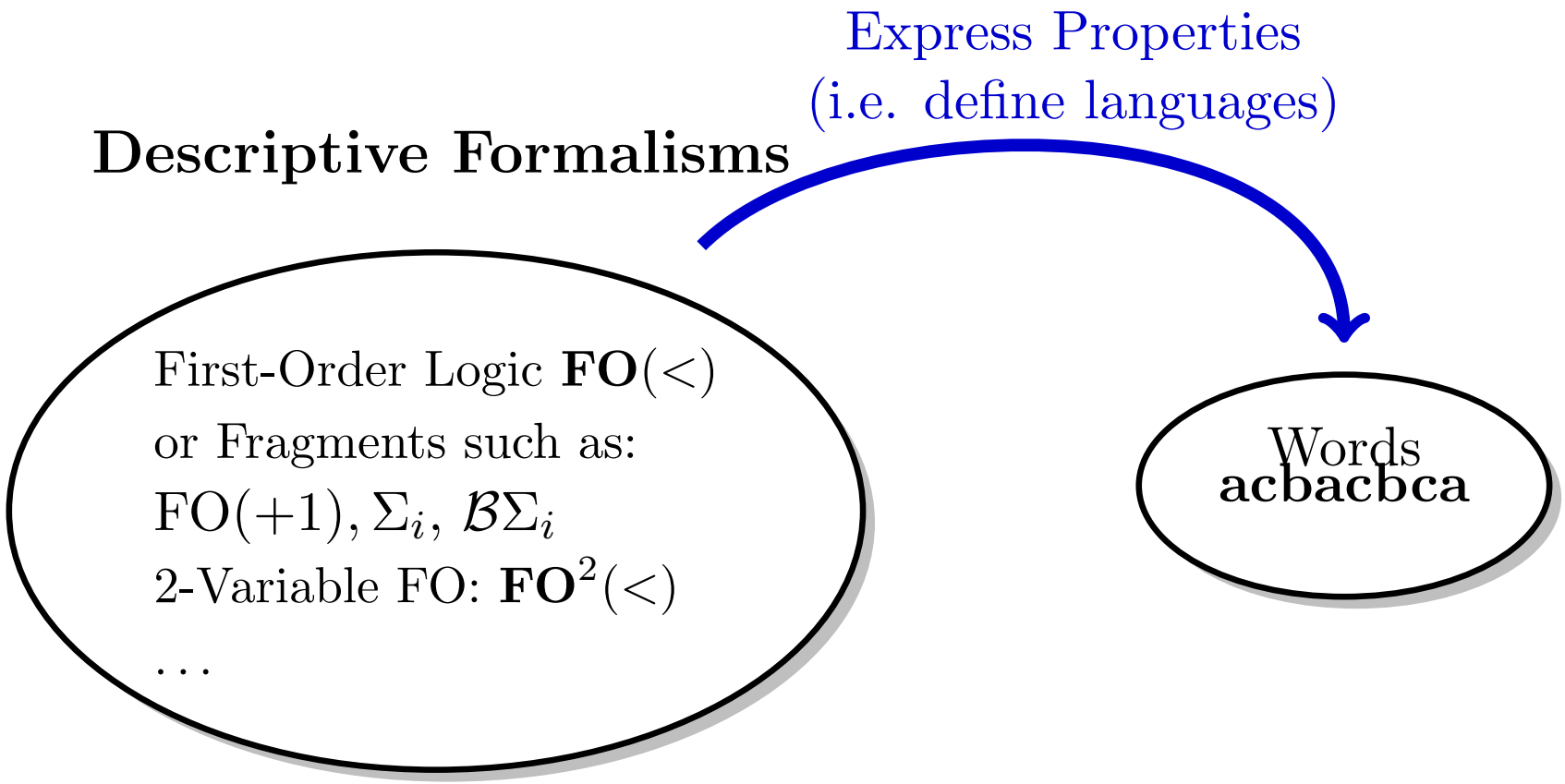


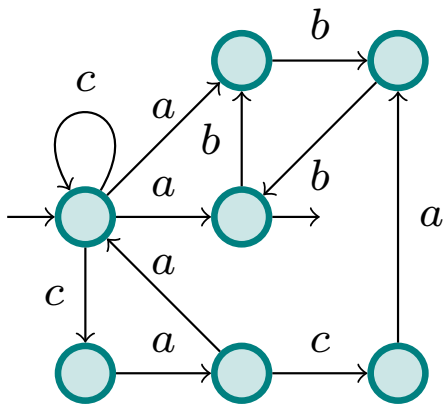
Objective: For each fragment, understand what it can express.

i.e. **What languages belong to the associated class $\mathcal{C}$?**

Methodology: The membership problem

Given such a class $\mathcal{C}$, the goal is to solve the associated **membership problem**:

L a regular language



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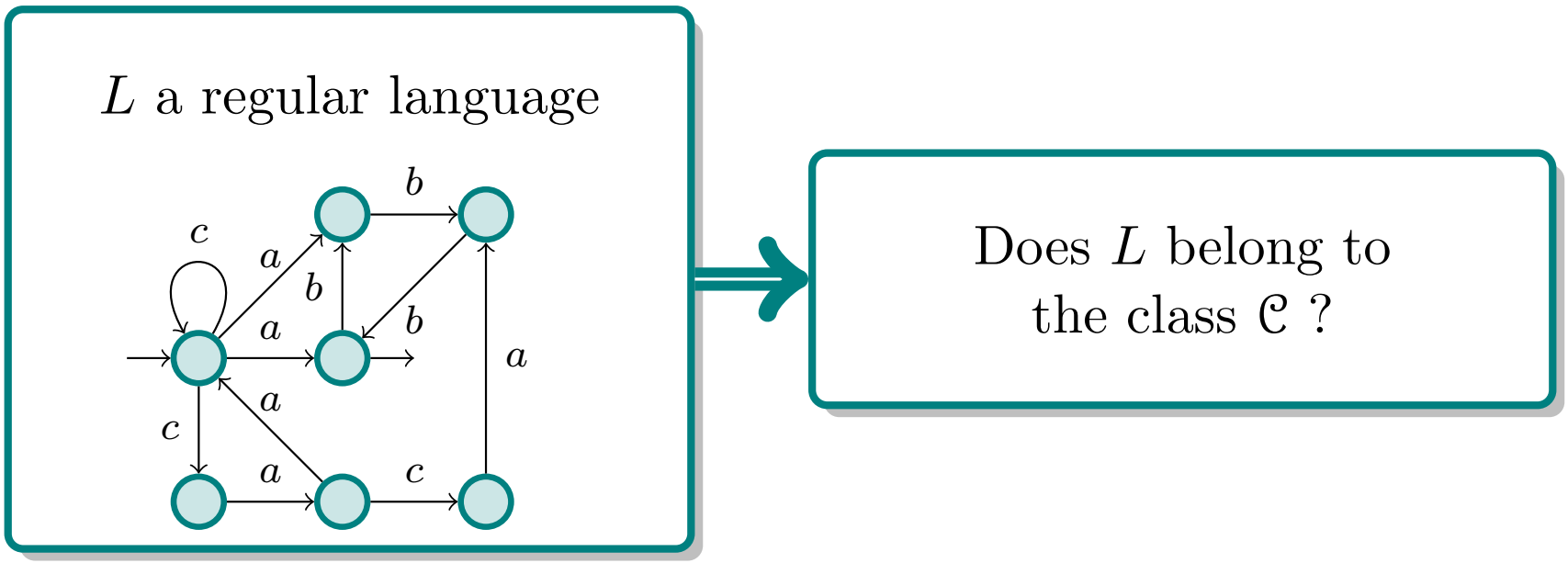

There are two stages to the problem:

- Stage 1: get an **algorithm** that decides it.
- Stage 2: find a generic way to **construct a sentence** witnessing membership of L in $\mathcal{C}$ when it exists.

Example - McNaughton-Papert-Schützenberger

Given a regular language L, the following properties are equivalent:

- L is definable in $\mathrm{FO}(<)$
- The minimal automaton of L is counter-free
- The syntactic monoid of L is aperiodic

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Why is it interesting ?

1. The theorem itself is an effective description of the class FO($<$).
2. The proofs are constructive: if we have the minimal automaton in hand, we can **construct a sentence** for L by induction.
   $\Rightarrow$ We get normal forms for FO($<$) sentences over words.

Altogether, we learn a lot about FO($<$) from this theorem

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  - For other structures: infinite words, finite trees.

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- **Successful methodology since the 70s**, reproduced
  - For other logical classes on words (eg, several restrictions of FO).
  - For other structures: infinite words, finite trees.
- Still, the methodology **fails for important classes**.

The big problem: quantifier alternation

Quantifer Alternation: Classifying Formulas

What is a simple formula ?

$$\forall x \ (a(x) \Rightarrow \exists y \ (b(y) \wedge (y < x)))$$

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$$\exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 \forall x_6 \forall x_7 \exists x_8 \ \varphi(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$$

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$\Rightarrow \Pi_2$ formula.

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$\Rightarrow \Sigma_7$ formula (φ quantifier-free)

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FO Quantifier Alternation Hierarchy

Level i: Σ_i

For all i, a Σ_i formula is (in prenex normal form)

$$\underbrace{\exists x_1, \ldots, x_{n_1} \forall y_1, \ldots, y_{n_2} \cdots\cdots}_{i \text{ blocks (starting with } \exists)} \quad \underbrace{\varphi(\bar{x}, \bar{y}, \ldots)}_{\text{quantifier-free}}$$

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Σ_i is not closed under complement $\Rightarrow$ we get two other classes:

Level i: Π_i

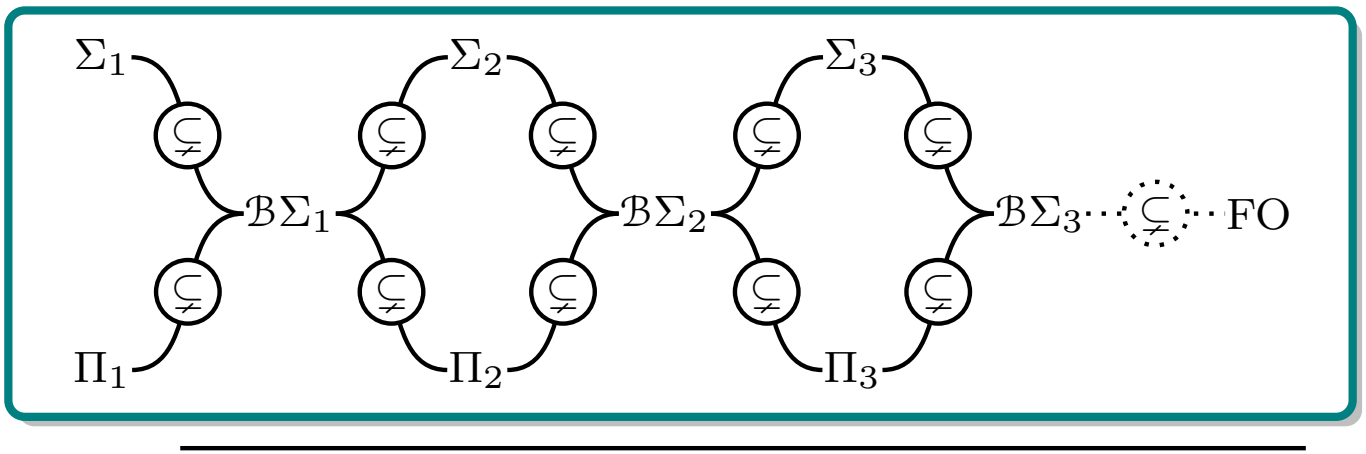
Negation of a Σ_i formula:

$$\underbrace{\forall x_1, \ldots, x_{n_1} \exists y_1, \ldots, y_{n_2} \cdots}_{i \text{ blocks (starting with } \forall)} \quad \varphi$$

Level i: $\mathcal{B}\Sigma_i$

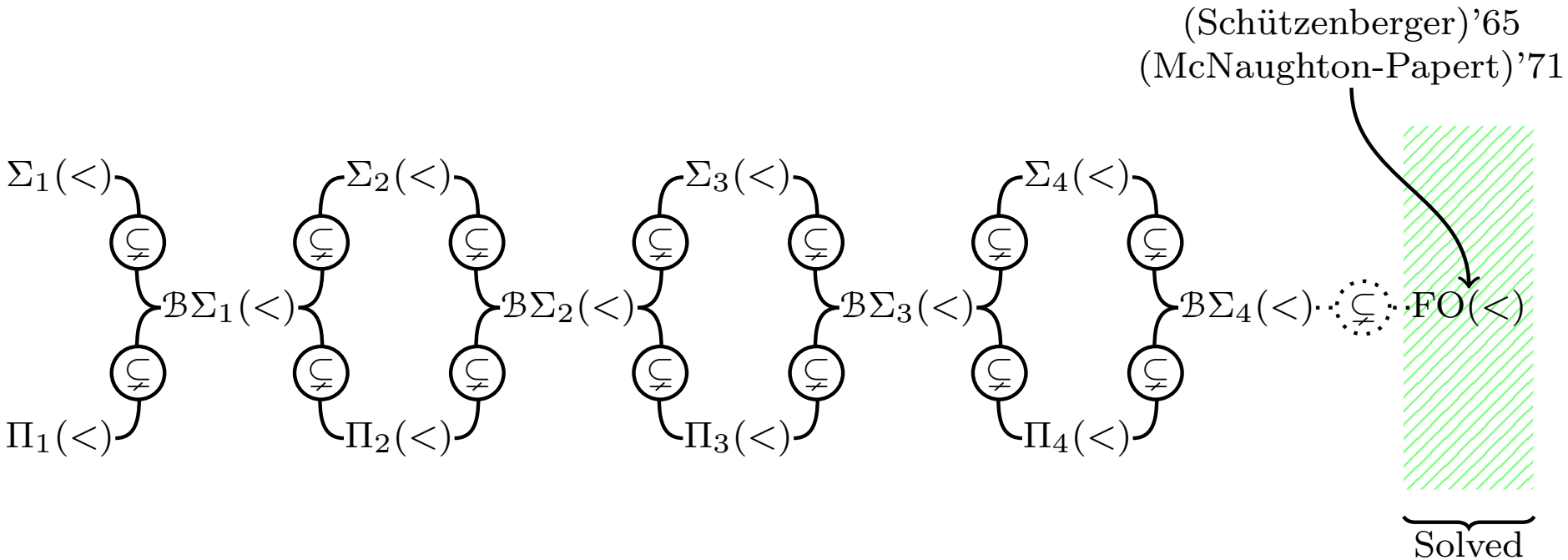
Boolean combinations of Σ_i (and Π_i) formulas.

FO(<) Quantifier Alternation Hierarchy: The Problem



Objective: get membership algorithms for all levels.

Quantifier Alternation: Membership state of the art



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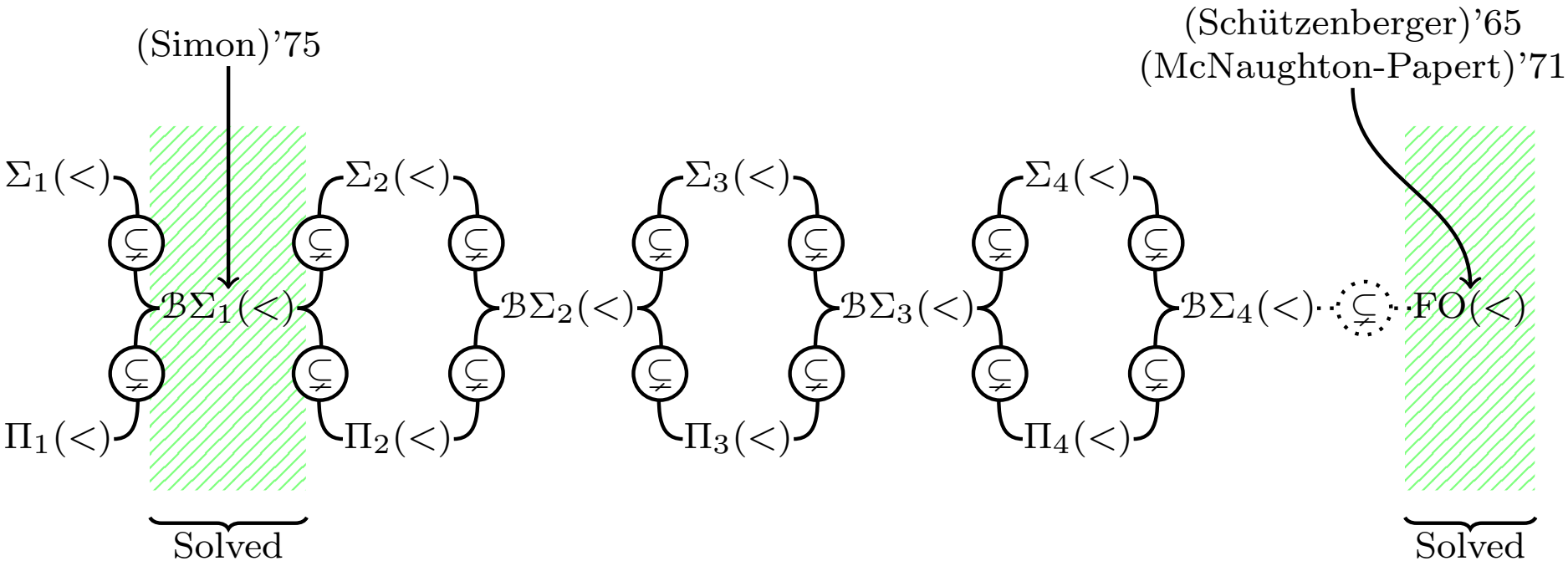

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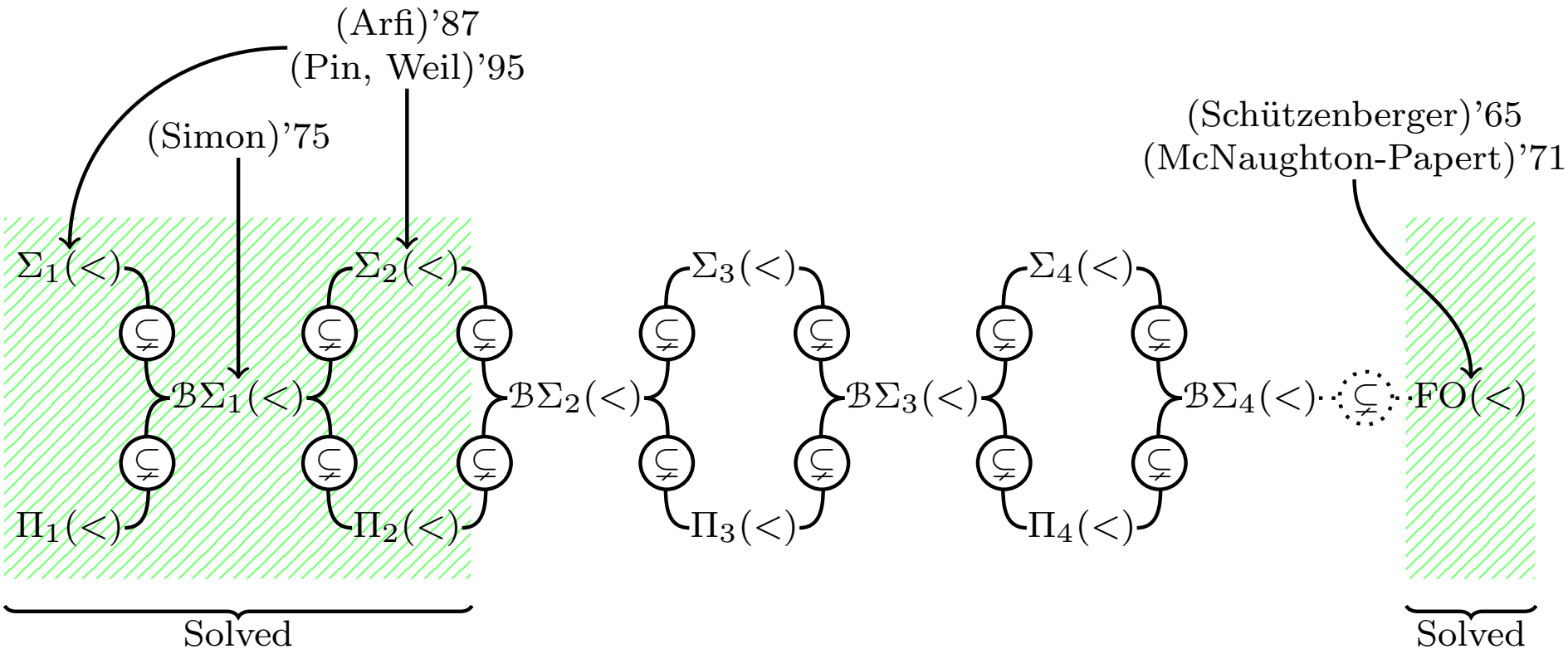

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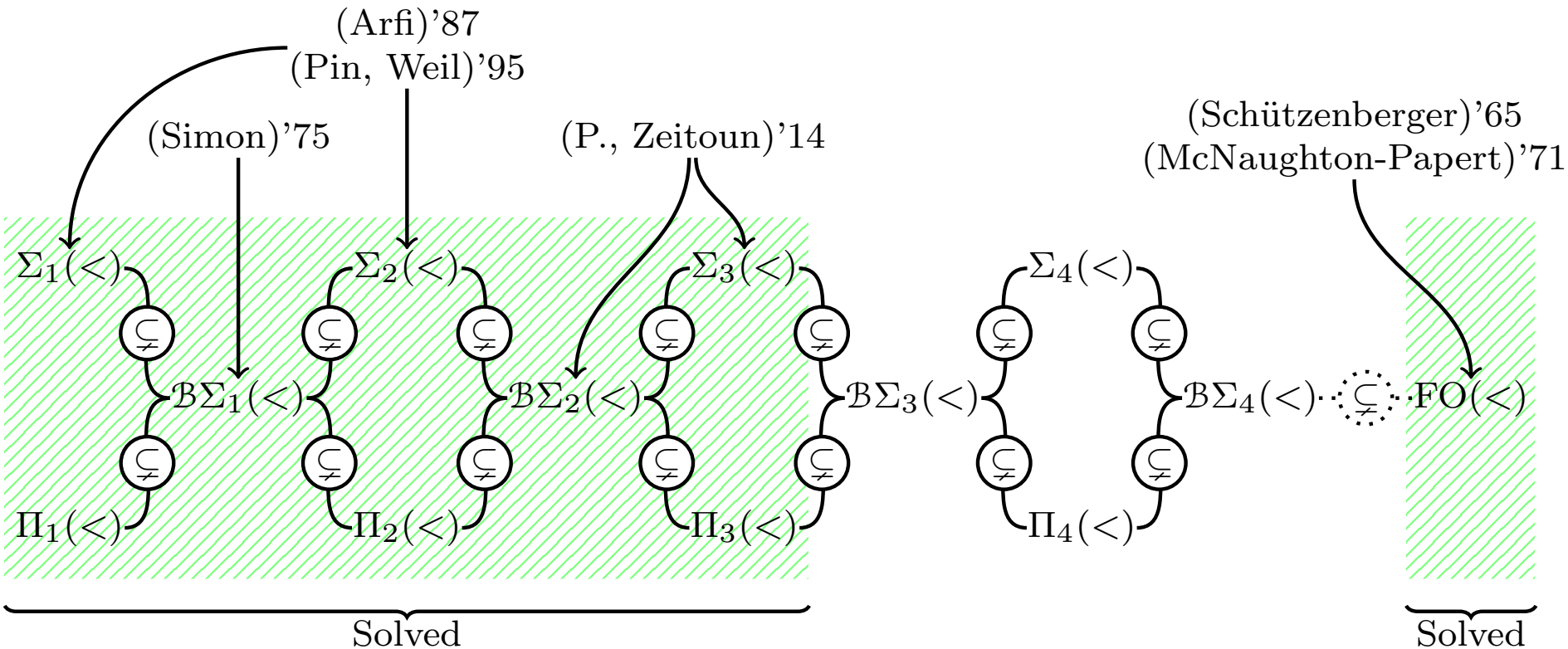

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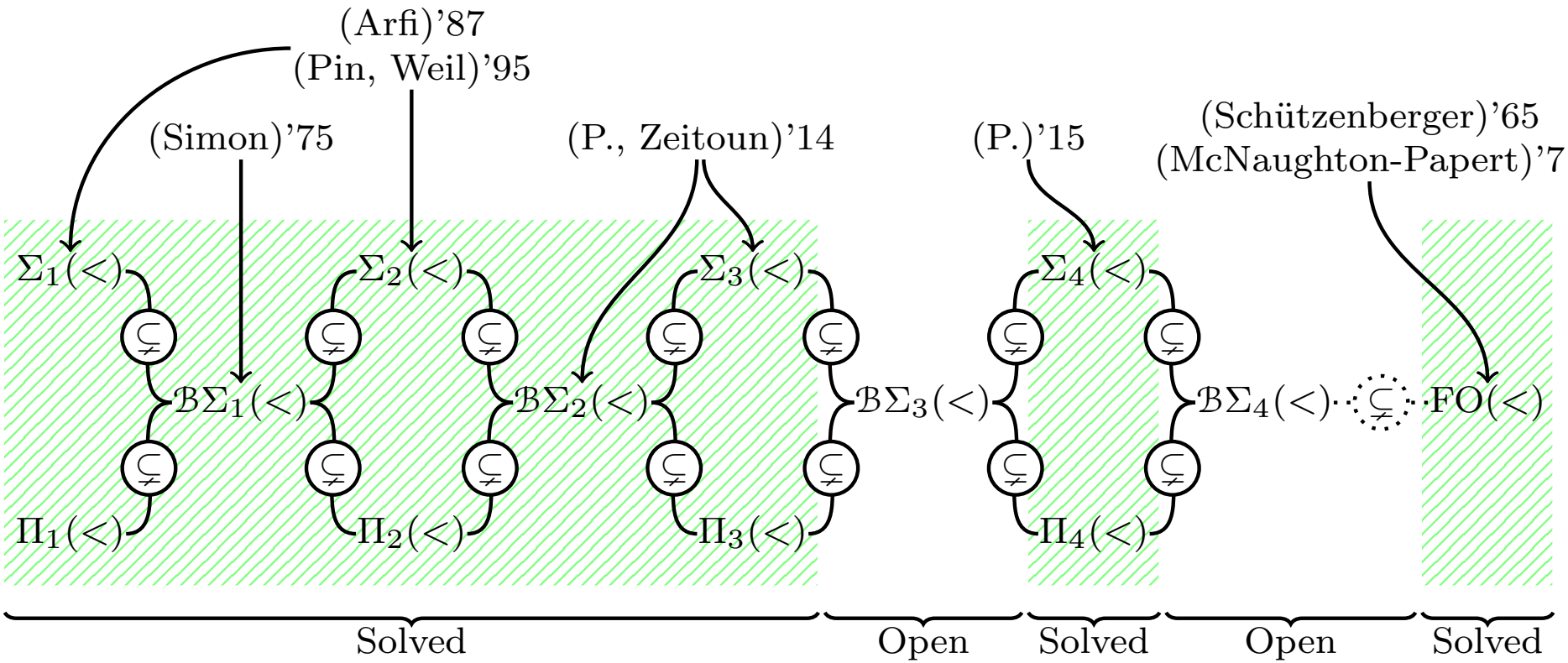

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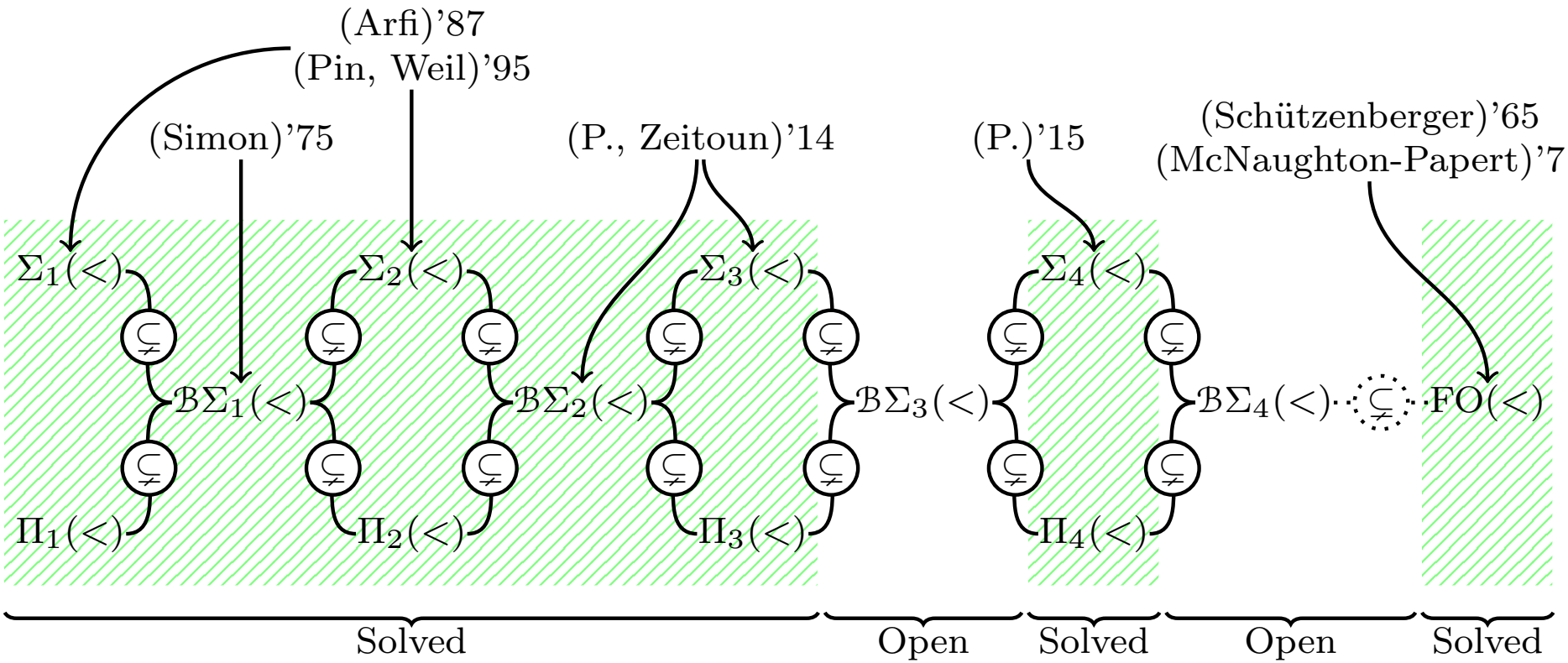

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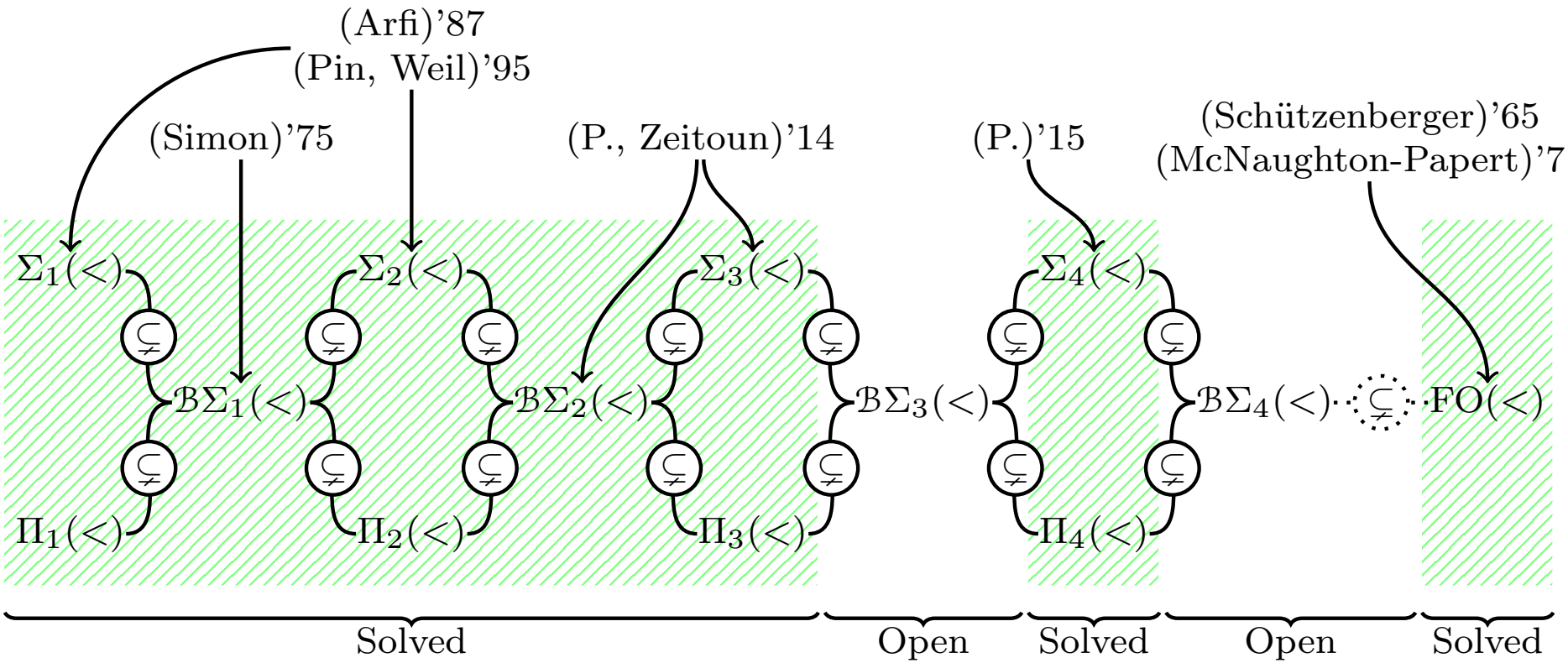


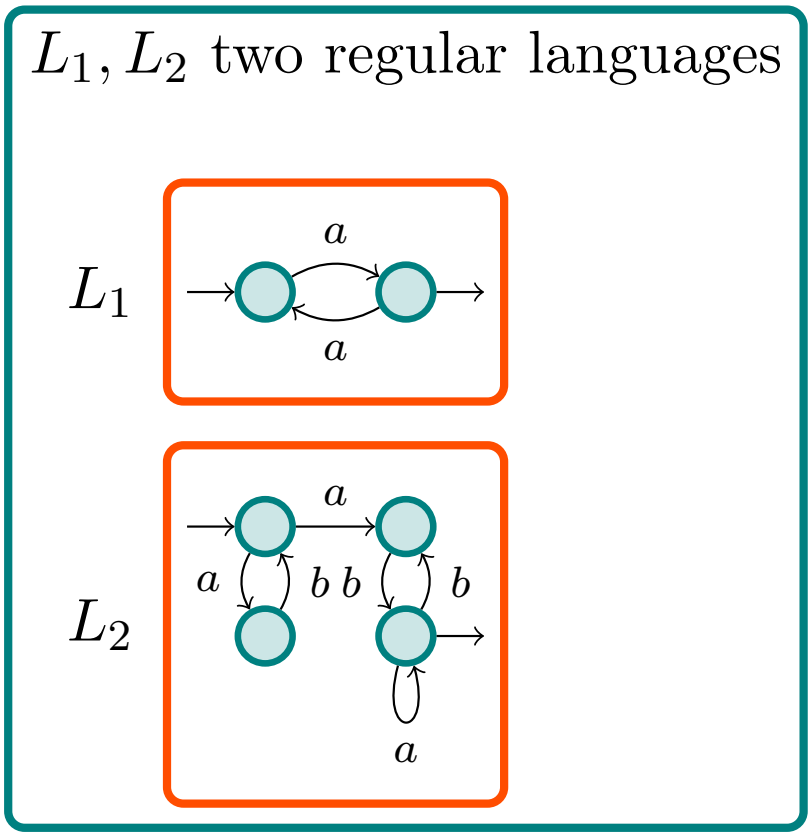
How are this results obtained ?

The previous slides only present a **third of the story** (at best).

The Separation Problem

Definition

Given a class of languages $\mathcal{C}$ (for example a level in the hierarchy), decide the following problem:



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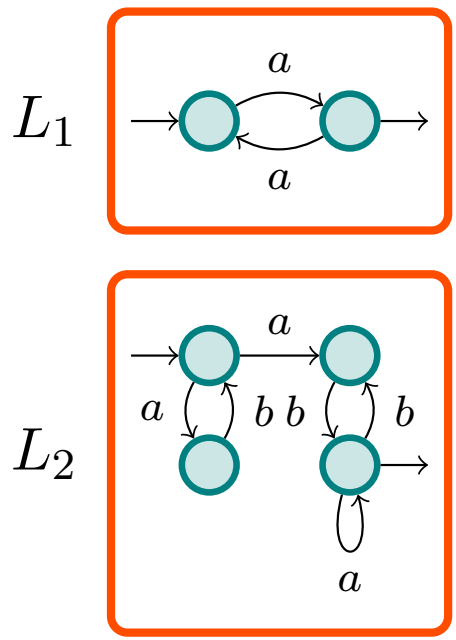
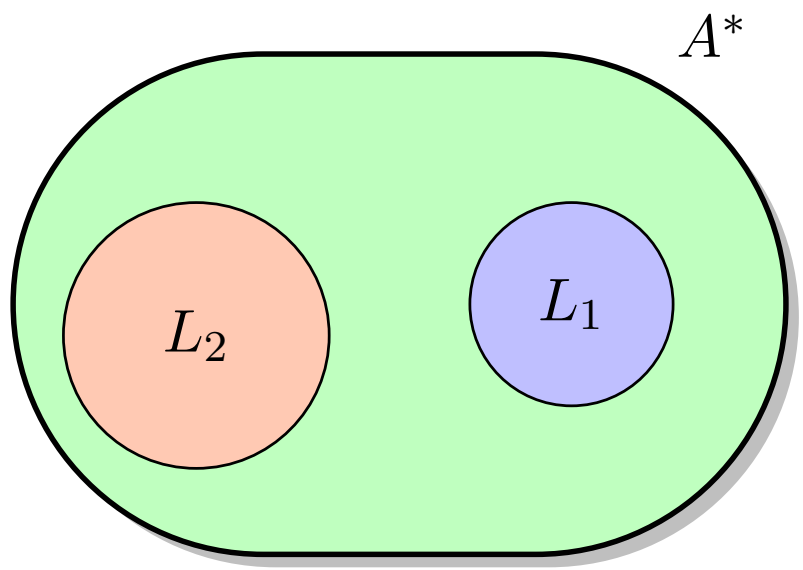
L_1, L_2 two regular languages



Can L_1 be separated from L_2 with a language of $\mathcal{C}$?



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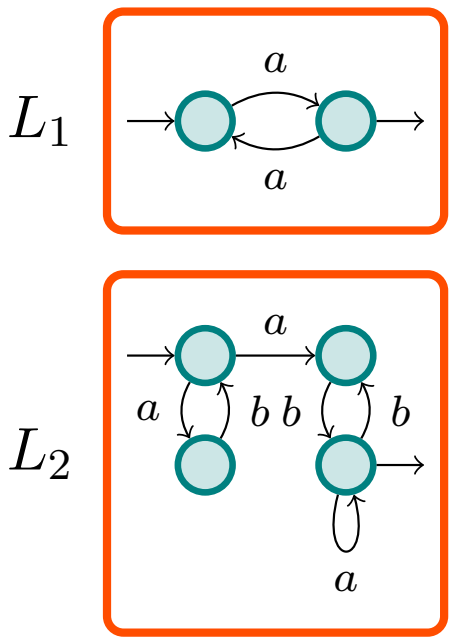

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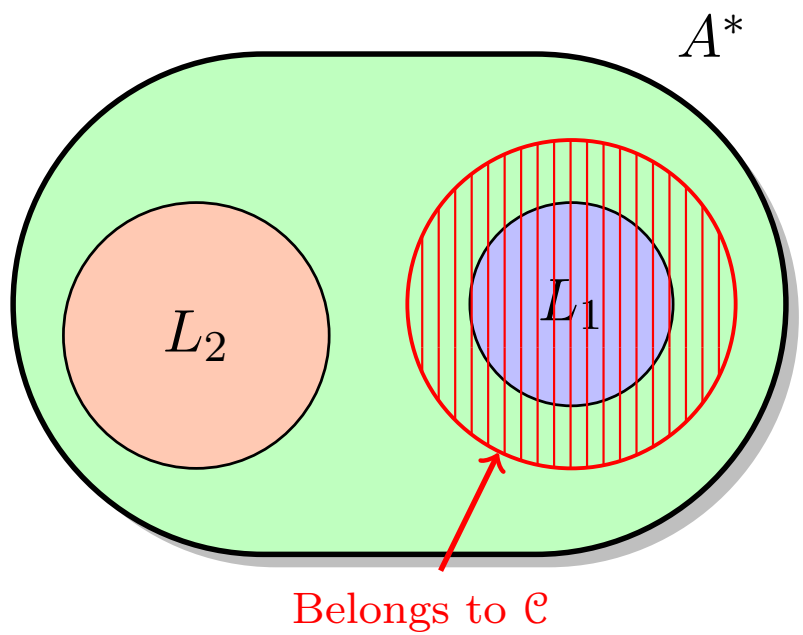

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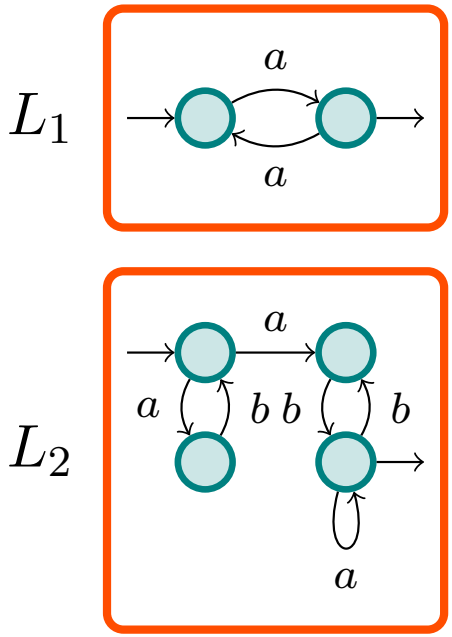

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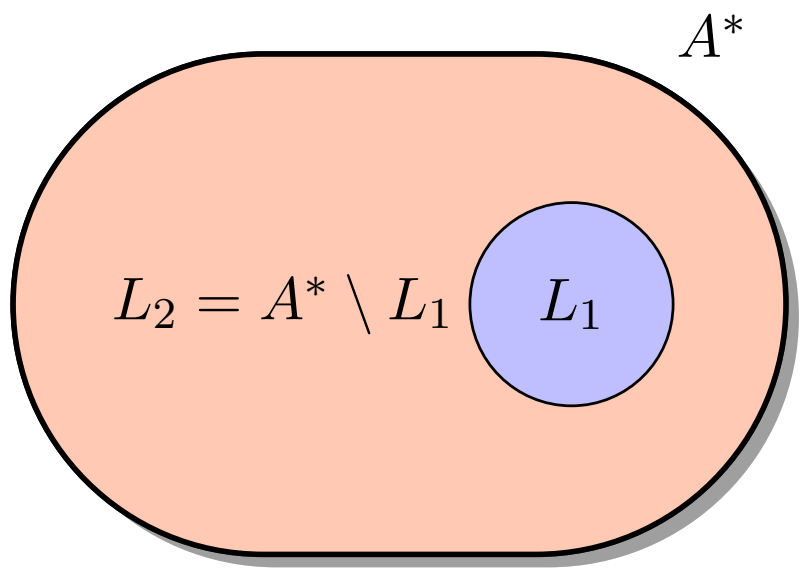

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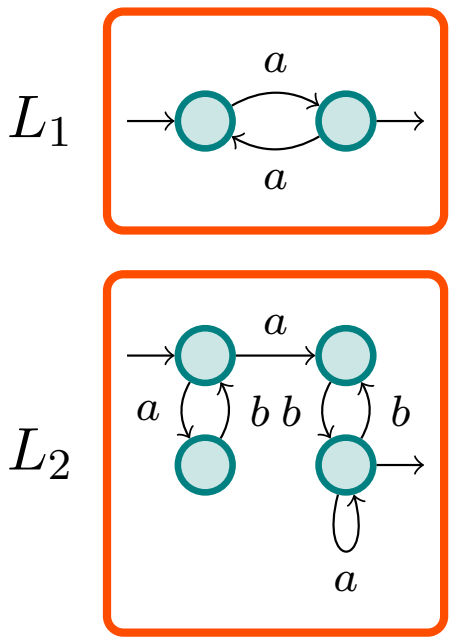

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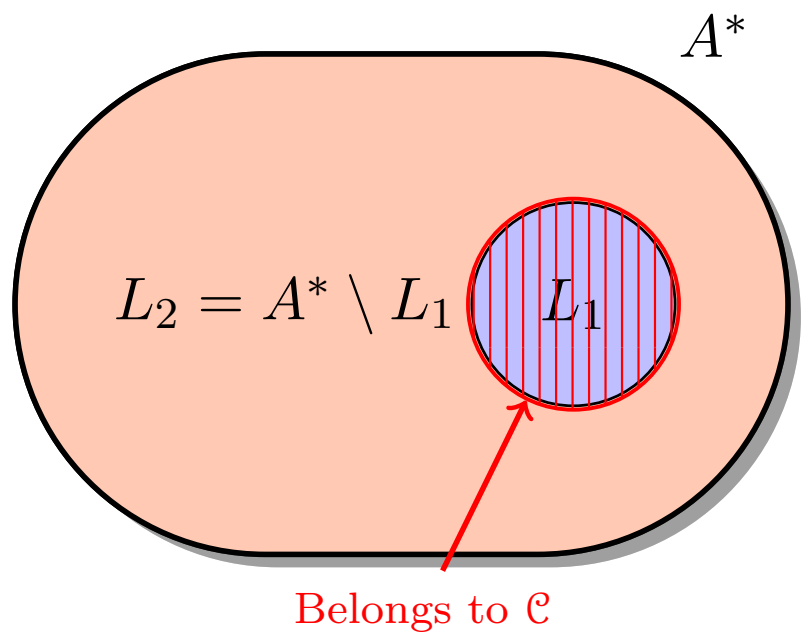

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Positive aspects:

- More rewarding with respect to the investigated class.
- Membership for $\mathcal{C}$ = Techniques applying to languages in $\mathcal{C}$ only. Separation for $\mathcal{C}$ = Techniques applying to **all languages**.

Membership for $\mathcal{C}$

Given a language L:

1. Does $L \in \mathcal{C}$?
2. If so, compute a description of L in $\mathcal{C}$.

Separation for $\mathcal{C}$

Given two languages L_1, L_2:

1. Can we **approximate** L_1 with some $K \in \mathcal{C}$? (allowed approximations given by L_2)
2. If so, compute $K \in \mathcal{C}$ realizing this approximation.

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All results that we have today for the hierarchy are **based on separation** (or more general problems):

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Transfer theorem (P.,Zeitoun)'14

For all $n \geq 1$,

$$\Sigma_n\text{-separation decidable} \Rightarrow \Sigma_{n+1}\text{-membership decidable}$$

Important Remark

Separation is **harder** than membership. The above above does **not** solve the whole hierarchy.

Transfer theorem: Σ_{n-1}-separation $\Rightarrow$ Σ_n-membership

Notation, for two states p, q: $L_{p,q} = \{w \mid p \xrightarrow{w} q\}$

Forbidden Patterns and Separation

A regular language **is definable in Σ_n** iff its minimal automaton has **no pattern**:

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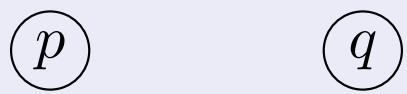

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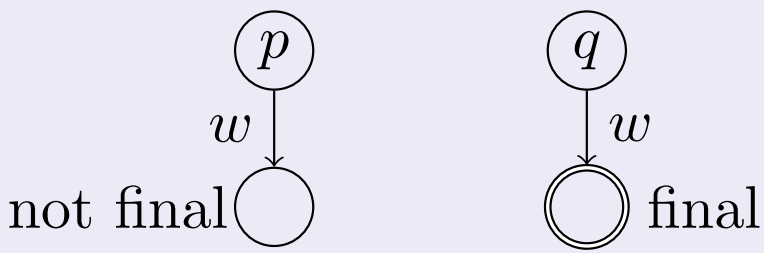

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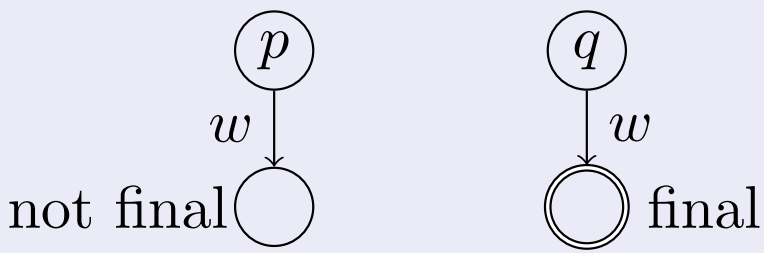

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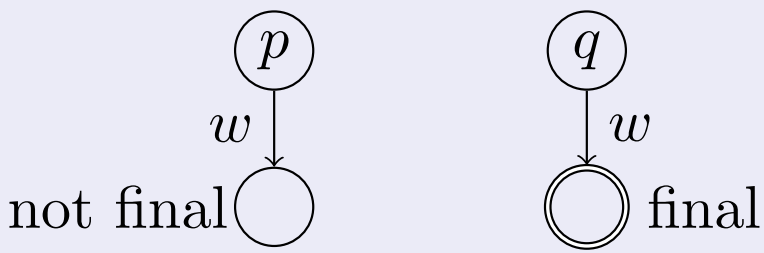

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Corollary

Solving **Σ_{n-1}-separation** yields a solution for **Σ_n-membership**.

Limits of this approach

We have the following:

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No similar result with separation on the right side.

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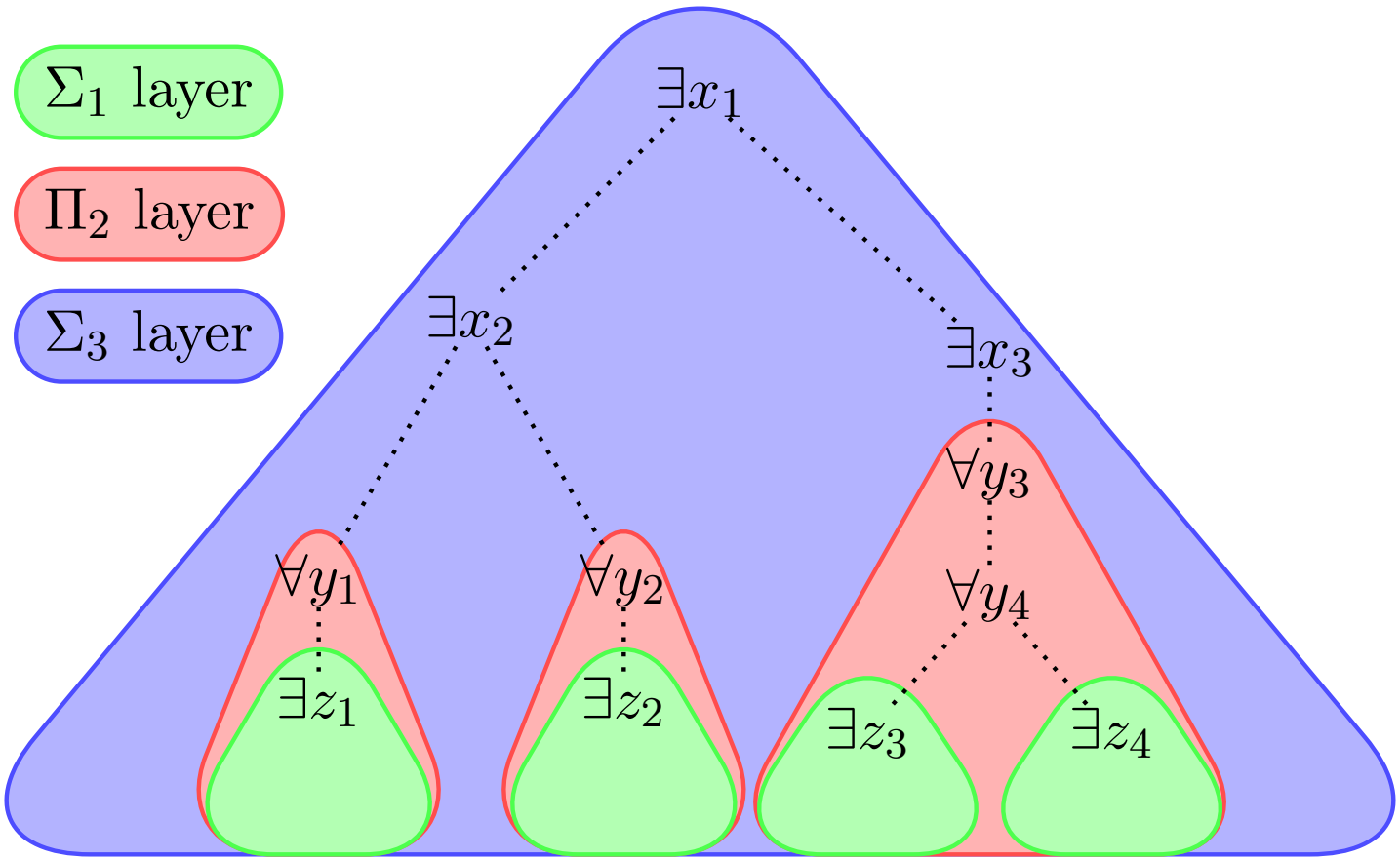
Let us explain why.

Hard part for both membership and separation:
Generic construction of **descriptions in $\mathcal{C}$**.

This is also the case for the transfer theorem.

Construction of Σ_n sentences

A Σ_n sentence is **layered**: Consider a Σ_3 sentence



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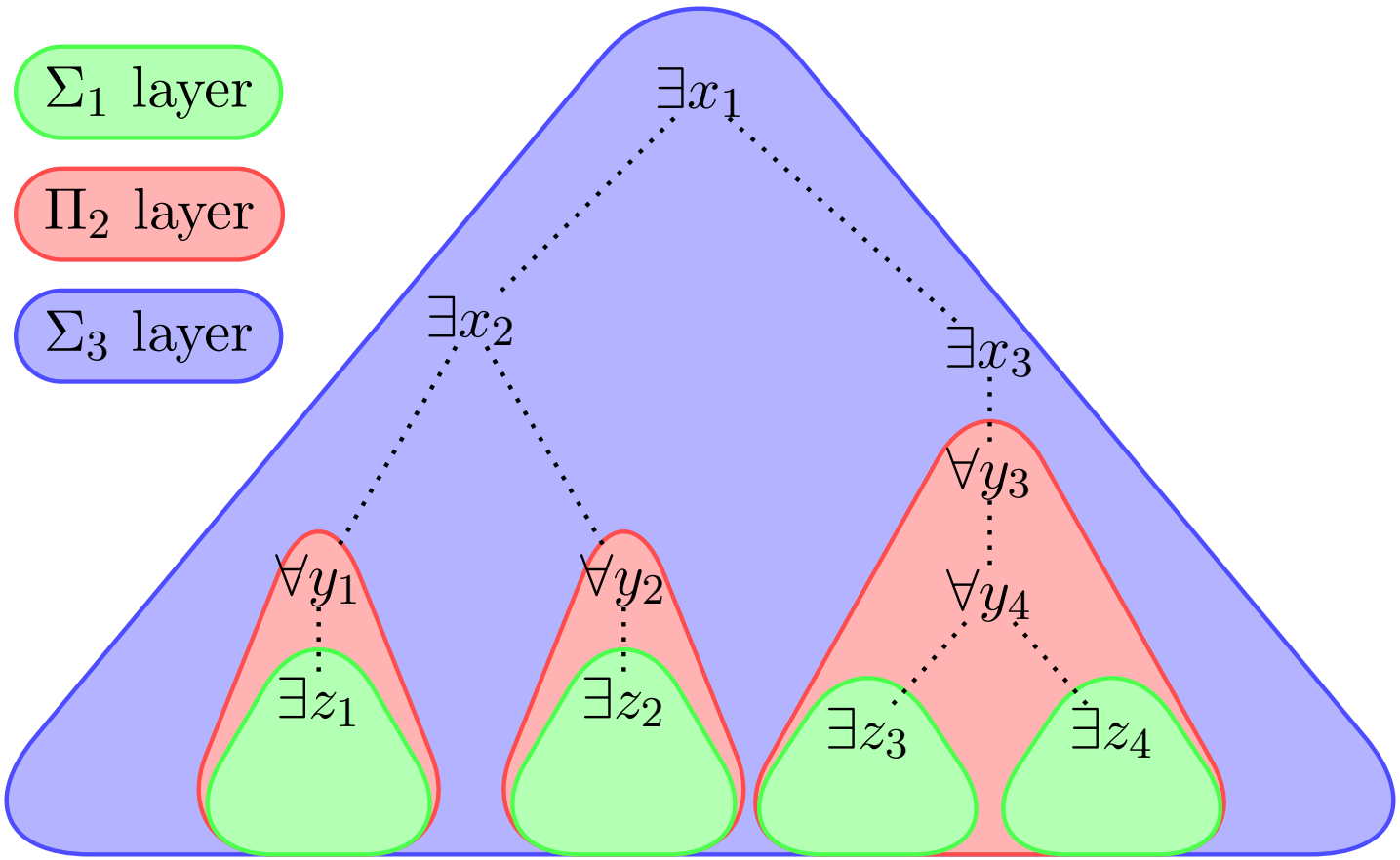

A generic construction should have several phases:
one for each layer

Construction of Σ_n sentences in the transfer theorem

Starting from a **DFA** $\mathcal{A}$ satisfying the transfer theorem, one builds a Σ_n sentence as follows:

- All languages needed for the **layers below** Σ_{n-1} are Σ_{n-1}-separators of $L_{p,q}$ from $L_{p,p} \cap L_{q,q}$ for some states p, q of $\mathcal{A}$.
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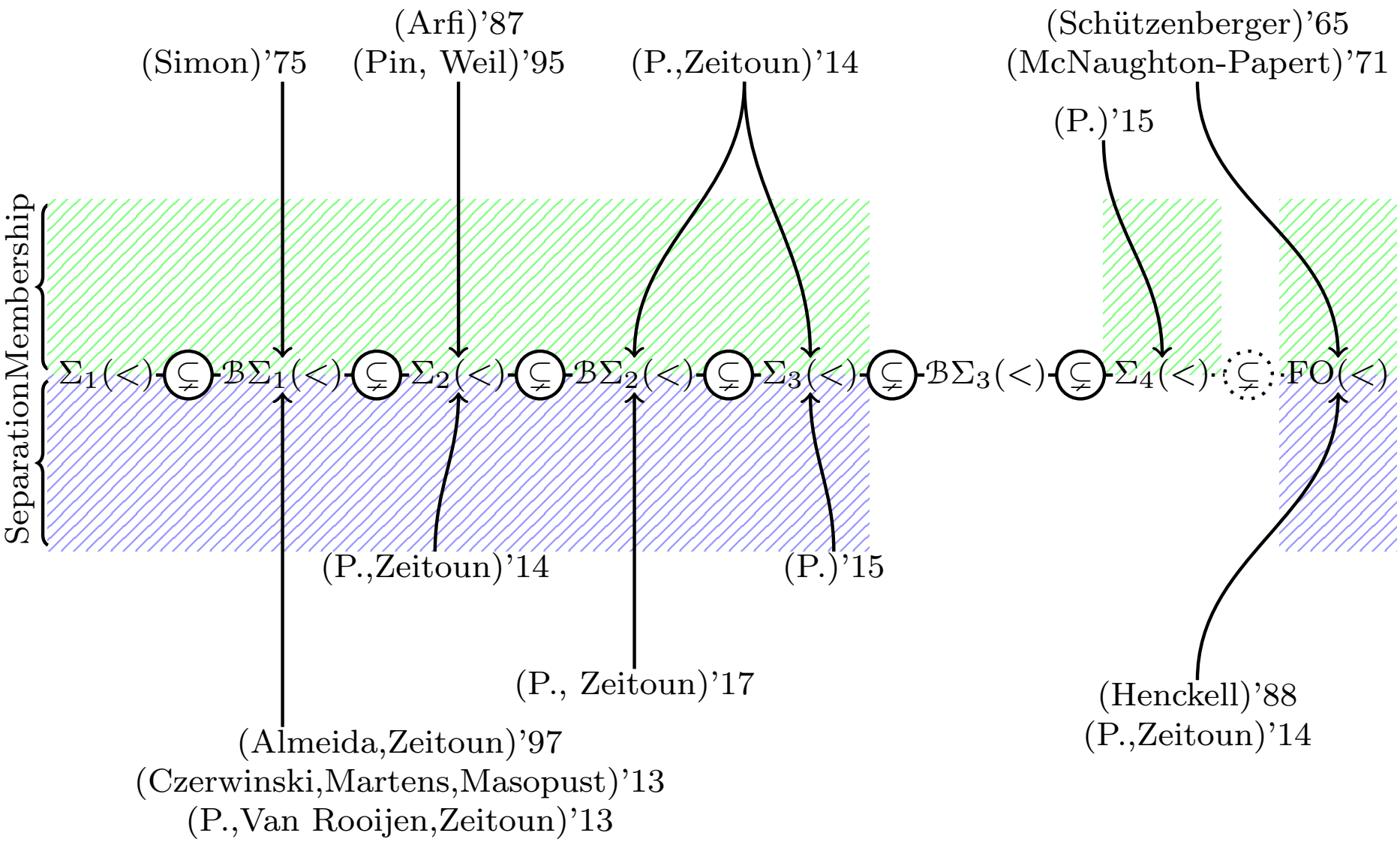
Separation is different: we do not have the Σ_n-layer in hand.
$\Rightarrow$ All layers must be considered simultaneously.

Current state of the art: Separation



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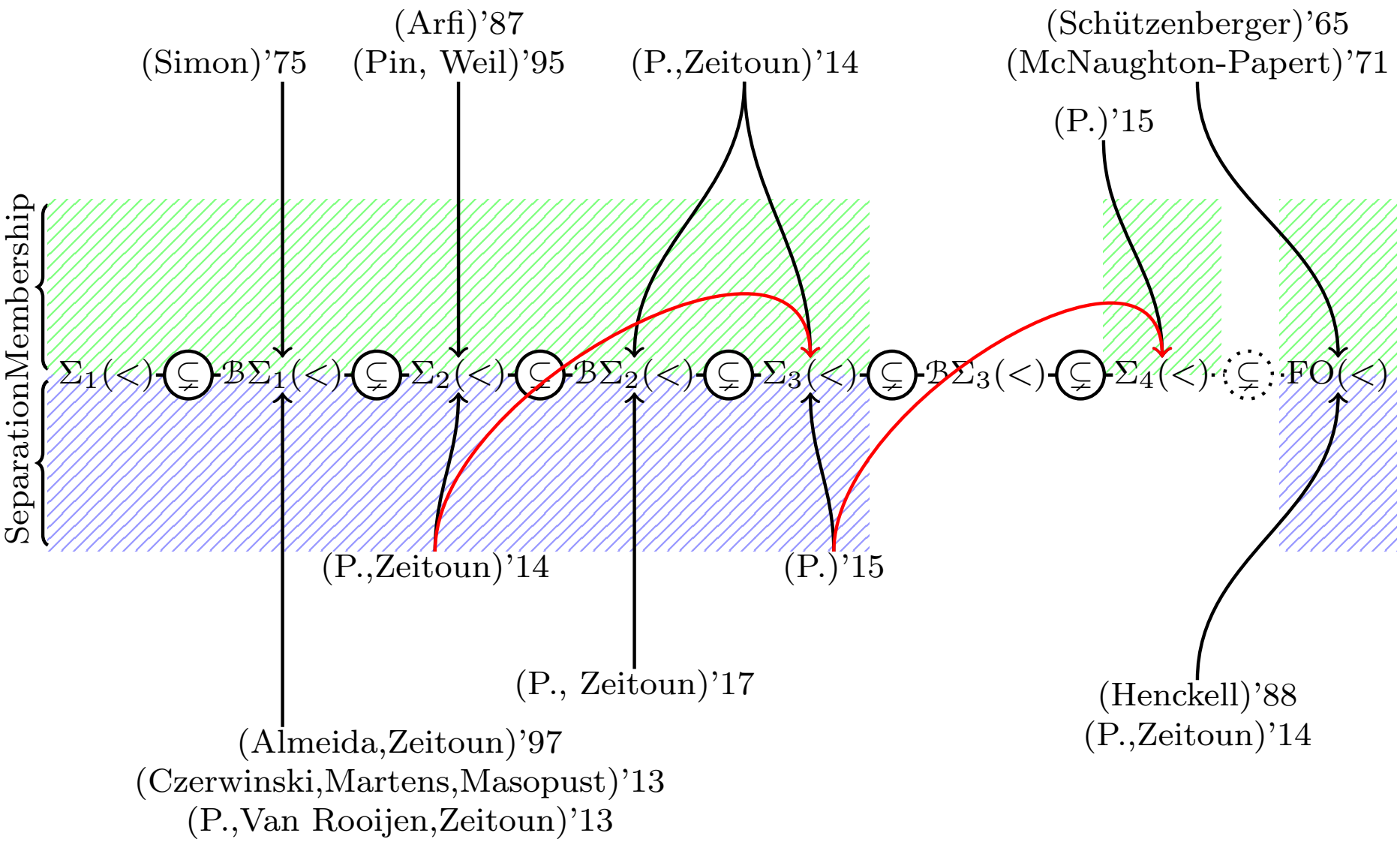

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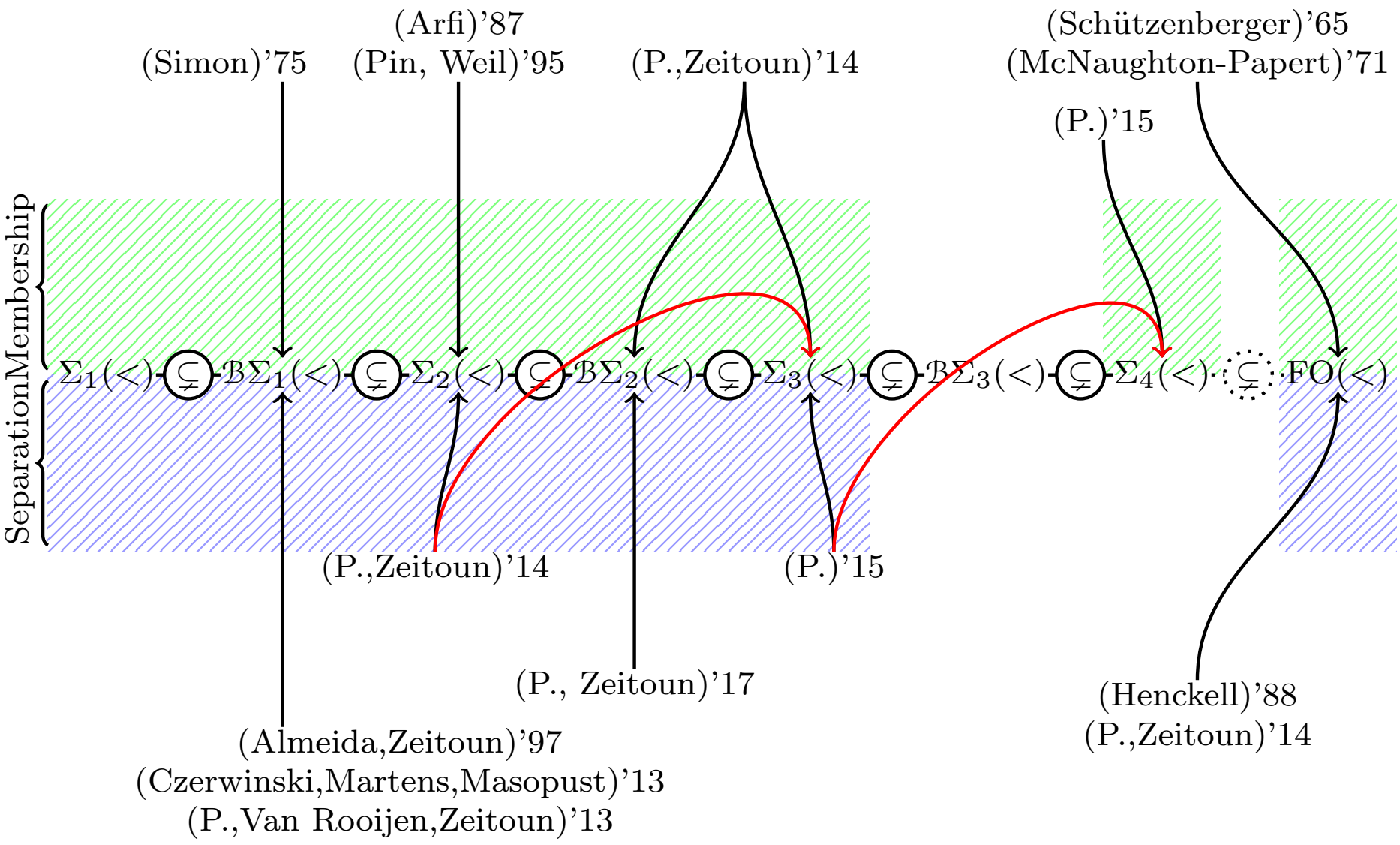

We are still missing **one third of the story**.

Concatenation hierarchies

Star-free languages (1)

McNaughton-Papert'71

Given a regular language L, the following properties are equivalent:

- L may be defined by an $\mathrm{FO}(<)$ sentence.
- L is **star-free**.

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- Closed under **union**, **union** and **complement**.
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- Closed under **marked concatenation**:

$$\text{Given } a \in A \quad K, L, a \mapsto KaL$$

$\Rightarrow$ Corresponds to existential quantification in FO($<$).

$$\exists x \; a(x) \wedge \varphi_K^{<x}(x) \wedge \varphi_L^{>x}(x)$$

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This is also the case for classes in the quantifier alternation hierarchy of FO(<).

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Classifies the star-free languages into half and full levels:

$$0 \longrightarrow \tfrac{1}{2} \longrightarrow 1 \longrightarrow \tfrac{3}{2} \longrightarrow 2 \longrightarrow \tfrac{5}{2} \longrightarrow 3 \cdots\cdots$$

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$Pol(\mathcal{C})$ built by closing the class $\mathcal{C}$ under:

- Union ($\bigcup$).
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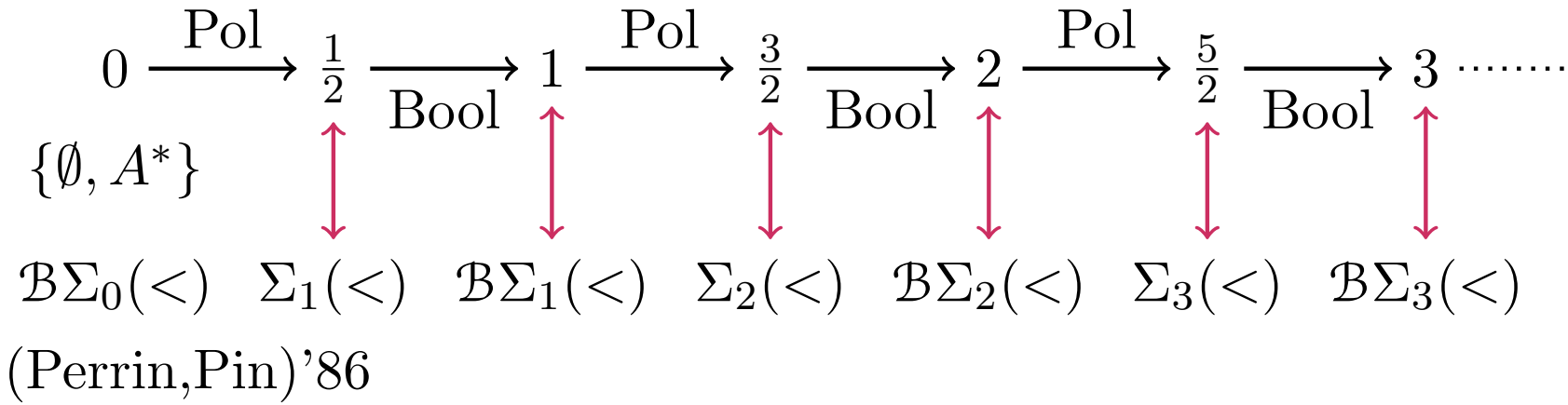

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Generic template: Concatenation hierarchies

Previous slide is an example of a **generic** construction.

Generic template: Concatenation hierarchies

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Before we explain how, let us further motivate the introduction of concatenation hierarchies.

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Given a basis $\mathfrak{C}$, we define a **variant FO($\mathfrak{C}$) of first-order logic** equipped with the following signature:

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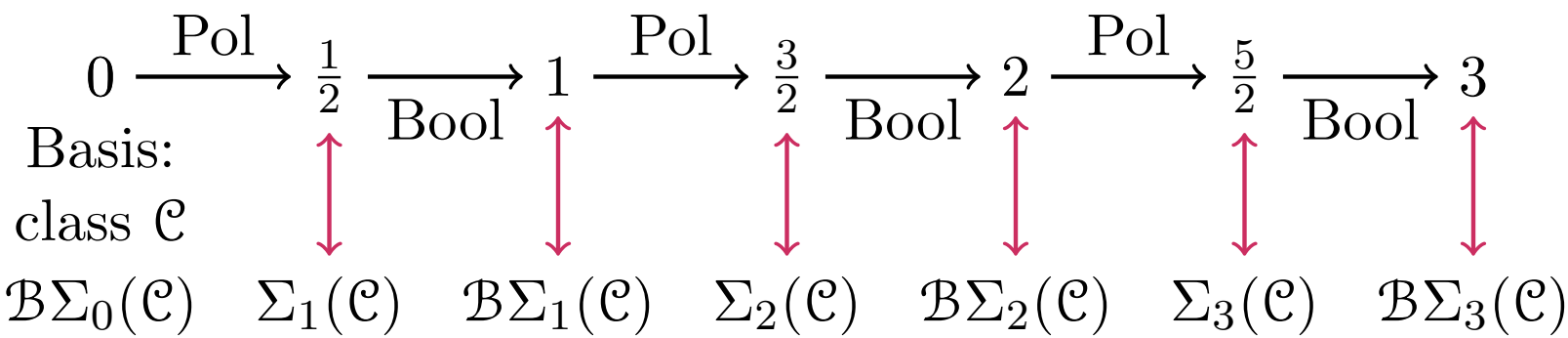

The logical connection is generic: Examples

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- $I_{A^*}(x, y)$ is $x < y$.
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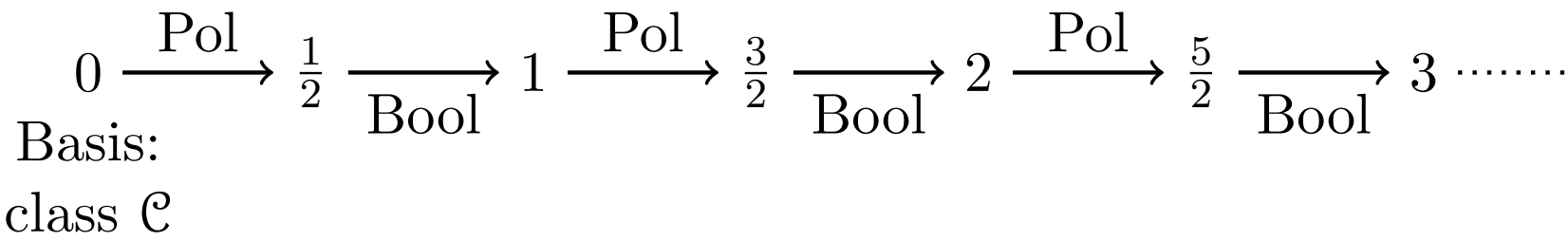
Generic separation results

Generic separation results (1)



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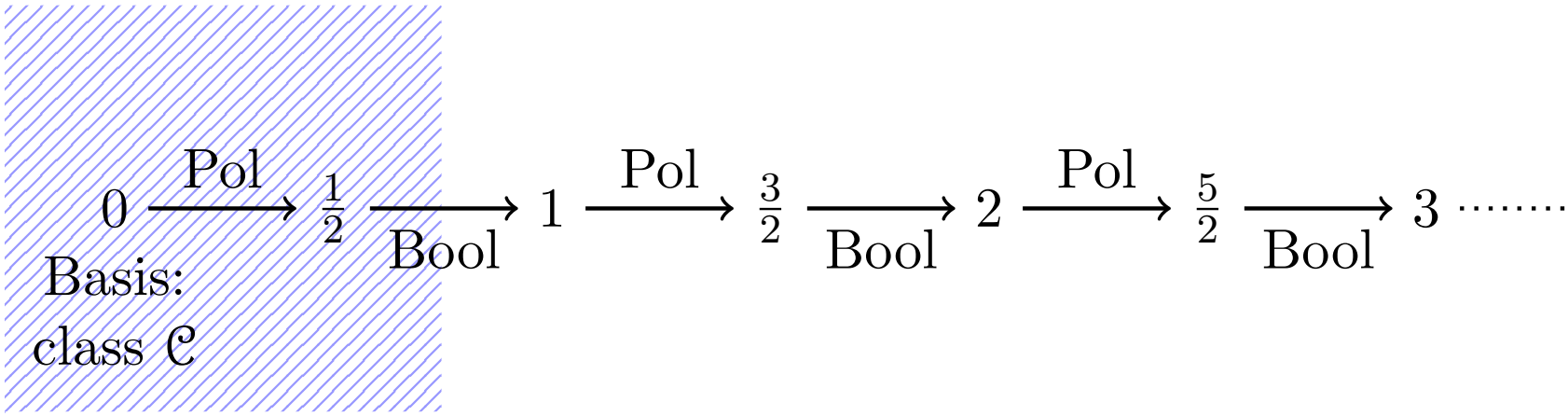

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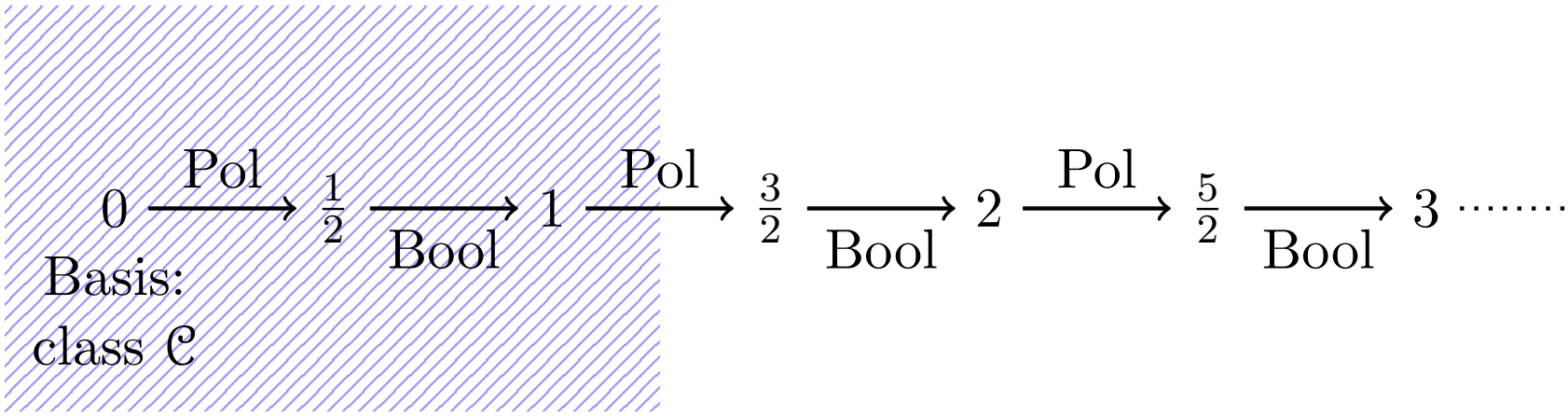

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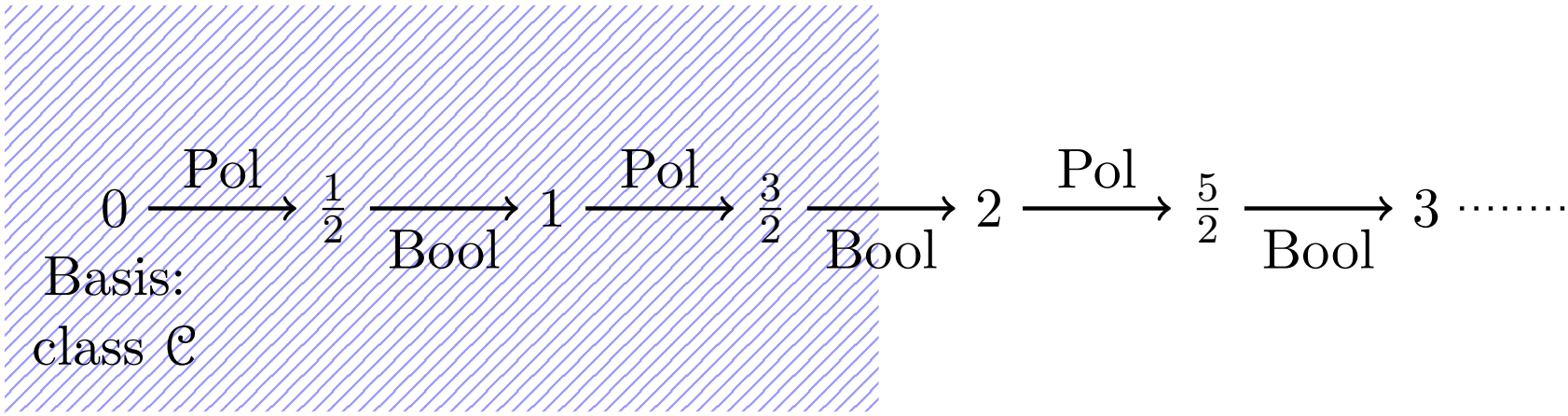

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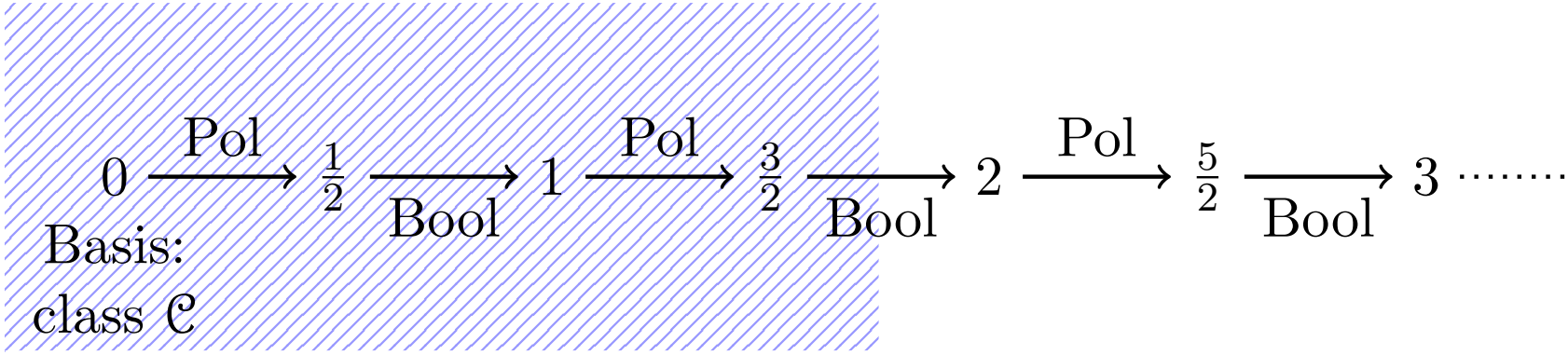

Advantages:

- These results treat many classes.
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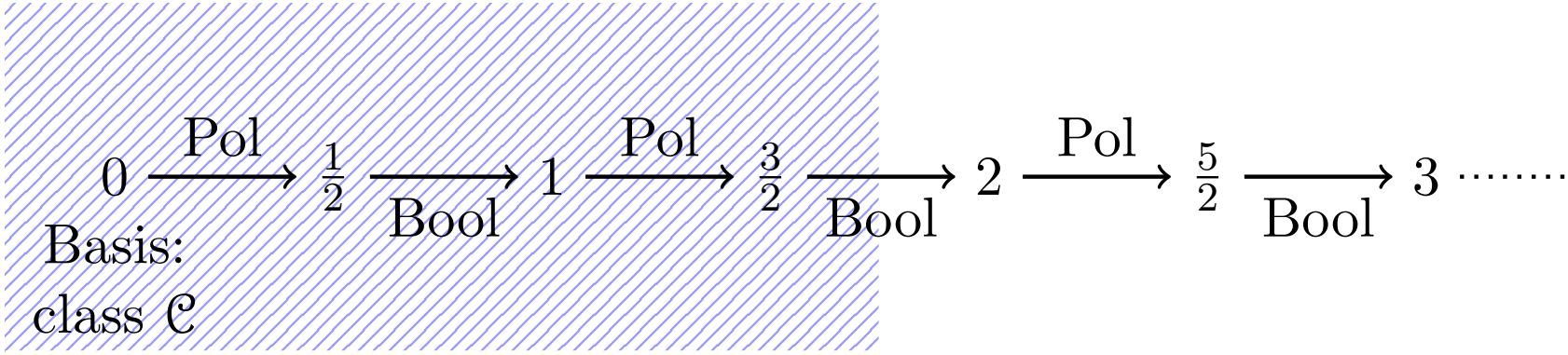

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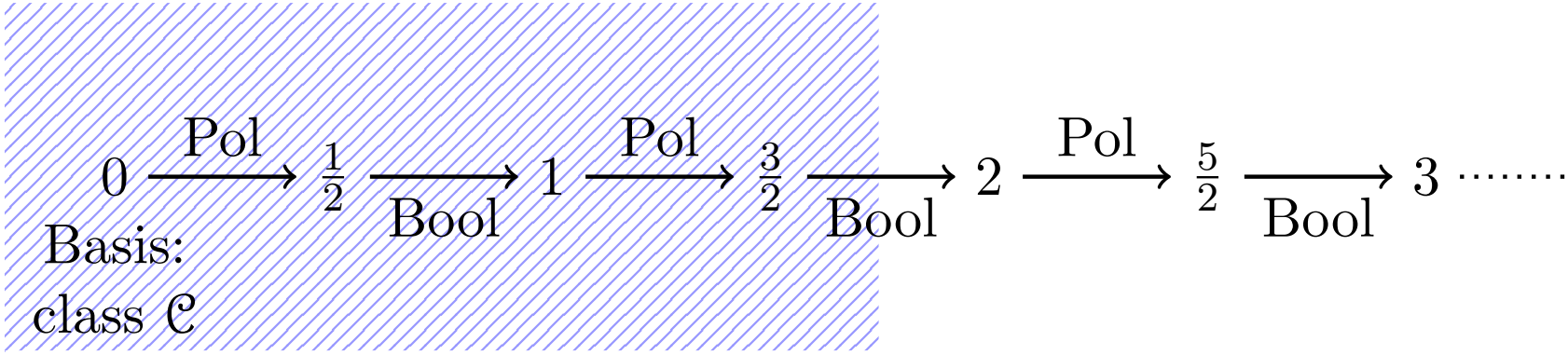
Generic separation results (2)



Advantages:

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Wait ! The speaker lied to us ! Refund !!!!!!

The results for FO(<) went **one full level higher**, didn't they ?

The almighty alphabet argument

Logical point of view (hierarchy within FO(<)):

$$\mathcal{B}\Sigma_0(<) \text{—} \Sigma_1(<) \text{—} \mathcal{B}\Sigma_1(<) \text{—} \Sigma_2(<) \text{—} \mathcal{B}\Sigma_2(<) \text{—} \Sigma_3(<) \text{—} \mathcal{B}\Sigma_3(<)$$

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Everything we know is captured by only **four generic results**:

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We now focus on $BPol(\mathcal{C})$**-separation**.

Separation for $BPol(\mathcal{C})$ when $\mathcal{C}$ is finite

$BPol(\mathcal{C})$-separation: Three main steps

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Warning

Many issues are hidden to simplify the presentation.

First step - preliminary remark

Our two closure operations have different properties:

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Consequence

Even if our goal is $BPol(\mathcal{C})$-separation, we prefer working with $Pol(\mathcal{C})$.

$BPol(\mathcal{C})$-separation: Three main steps (1)

Getting rid of Boolean closure

$Bool(\mathcal{D})$-separation $\Rightarrow$ generalized separation for $\mathcal{D}$

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Remark

When $n = 2$, we recover the classical notion.

Boolean closure theorem

For any two languages L_1, L_2 and $k \geq 1$, we denote by $(L_1, L_2)^k$, the tuple:

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Given a lattice $\mathcal{D}$ and two languages L_1, L_2, t.f.a.e.,

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Mission accomplished !

When $\mathcal{D} = Pol(\mathcal{C})$, this is a reduction from $BPol(\mathcal{C})$-separation to a problem for $Pol(\mathcal{C})$.

Boolean closure theorem: proof

We prove 2. $\Rightarrow$ 1. We have $k \geq 1$ such that $(L_1, L_2)^k$ is $\mathcal{D}$-separable.
We show by **induction on k** that L_1 is $Bool(\mathcal{D})$-separable from L_2

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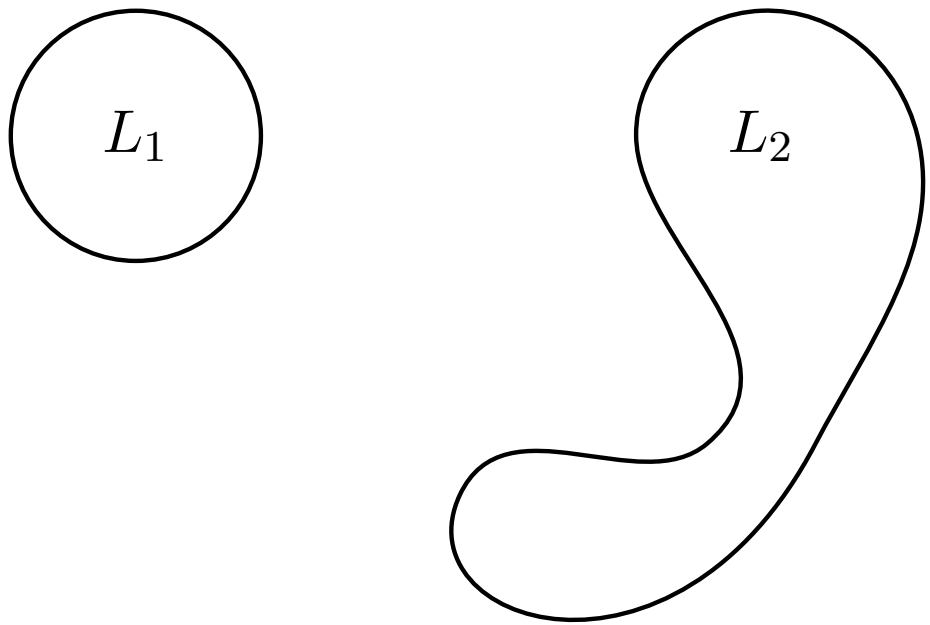

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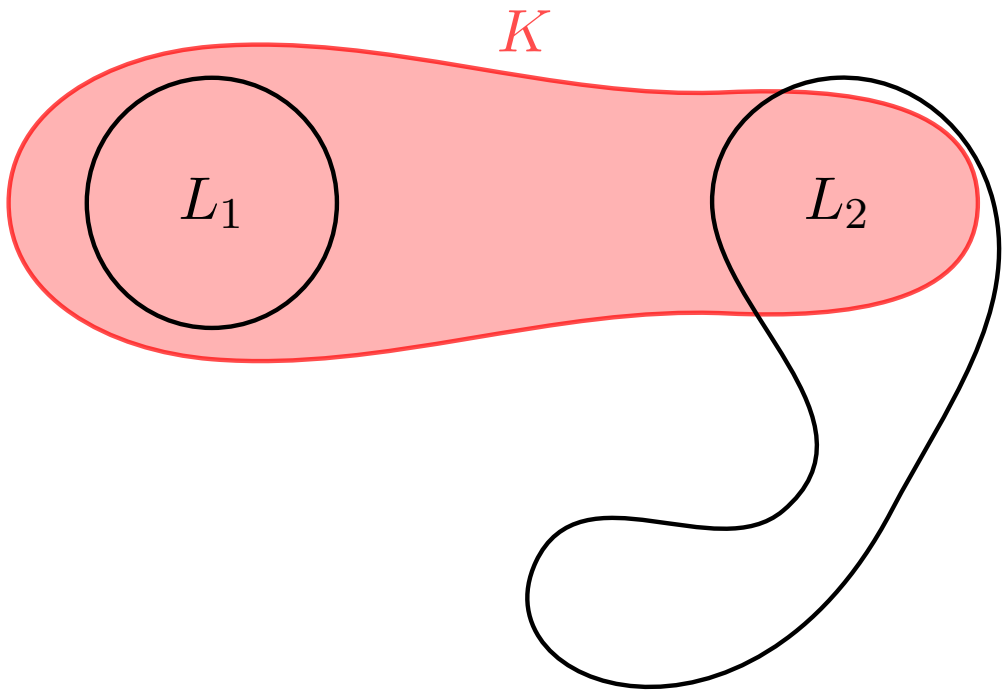

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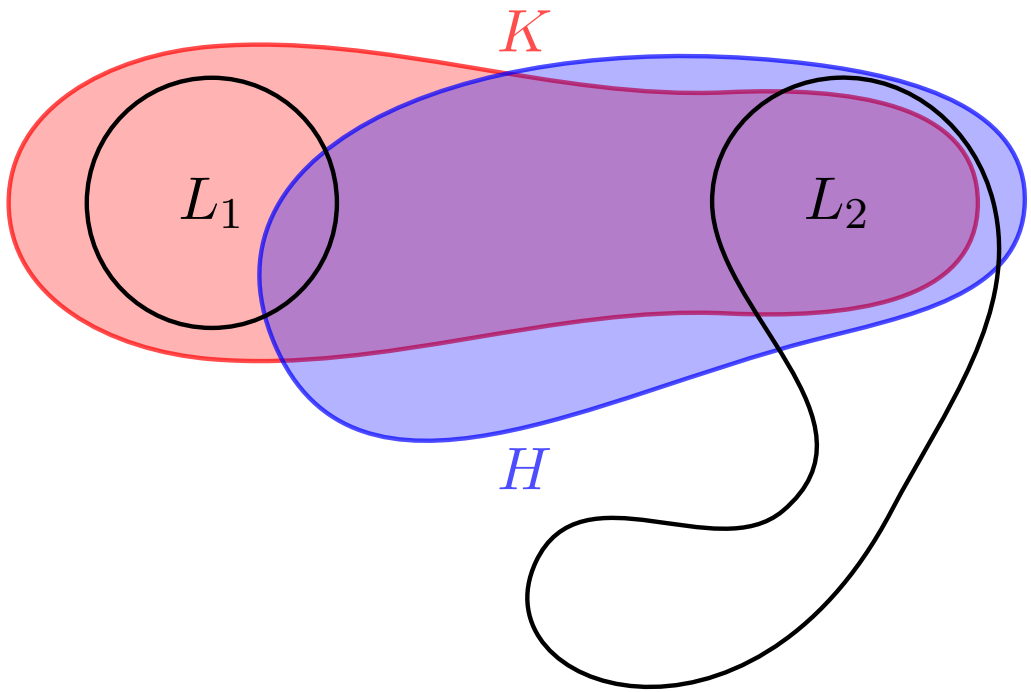

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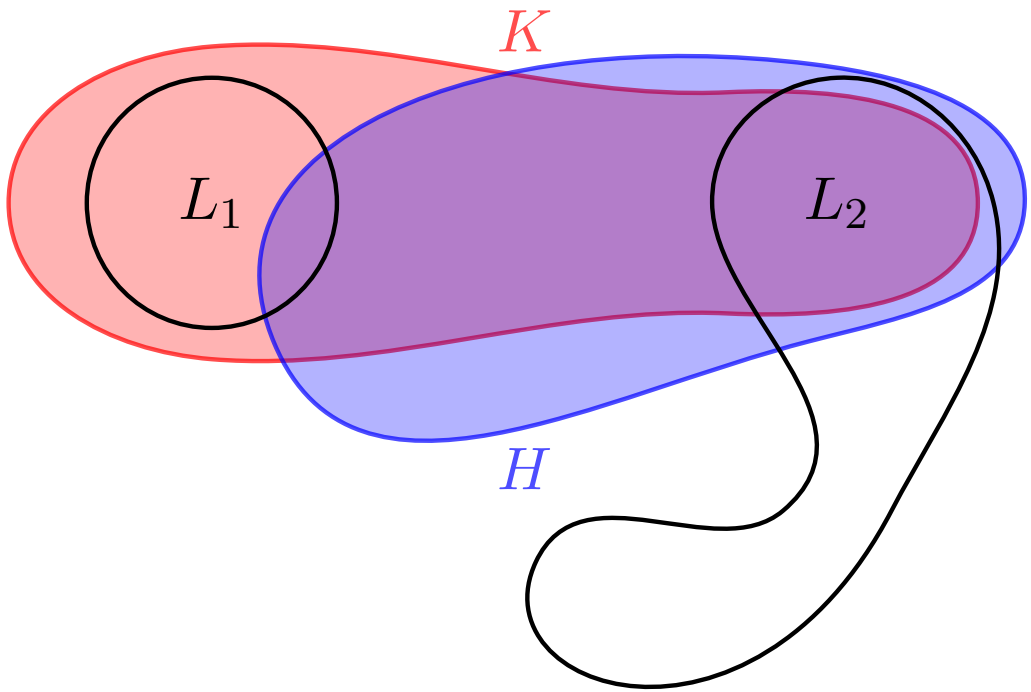

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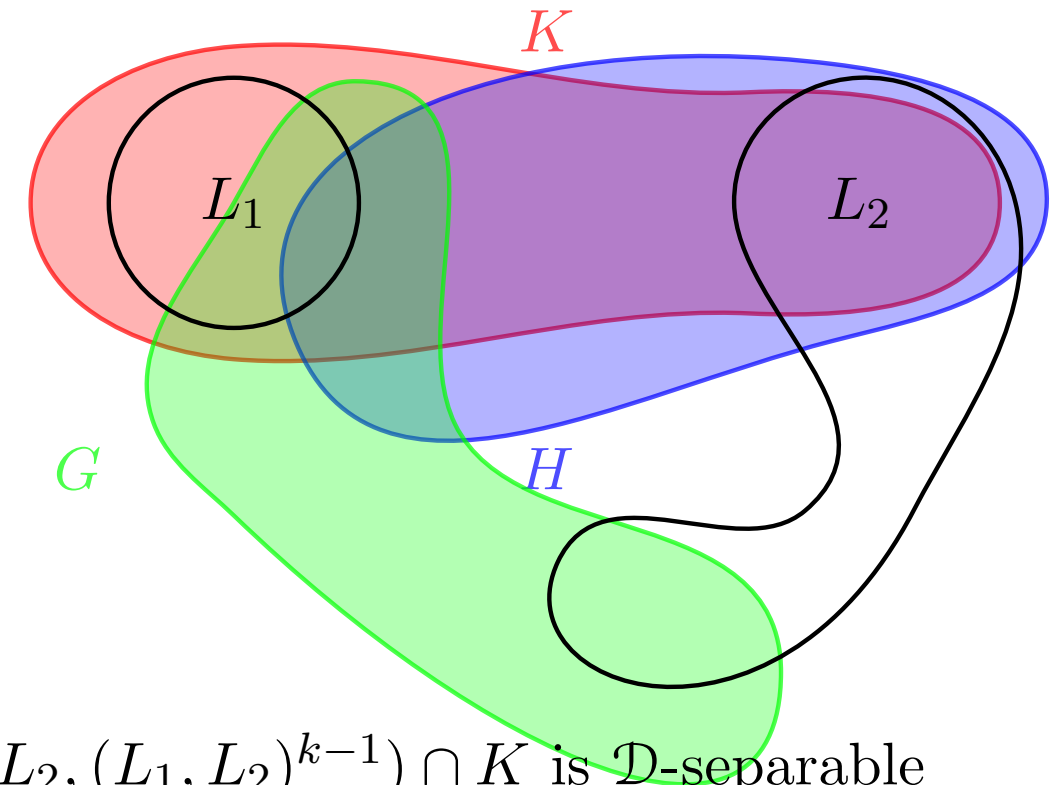

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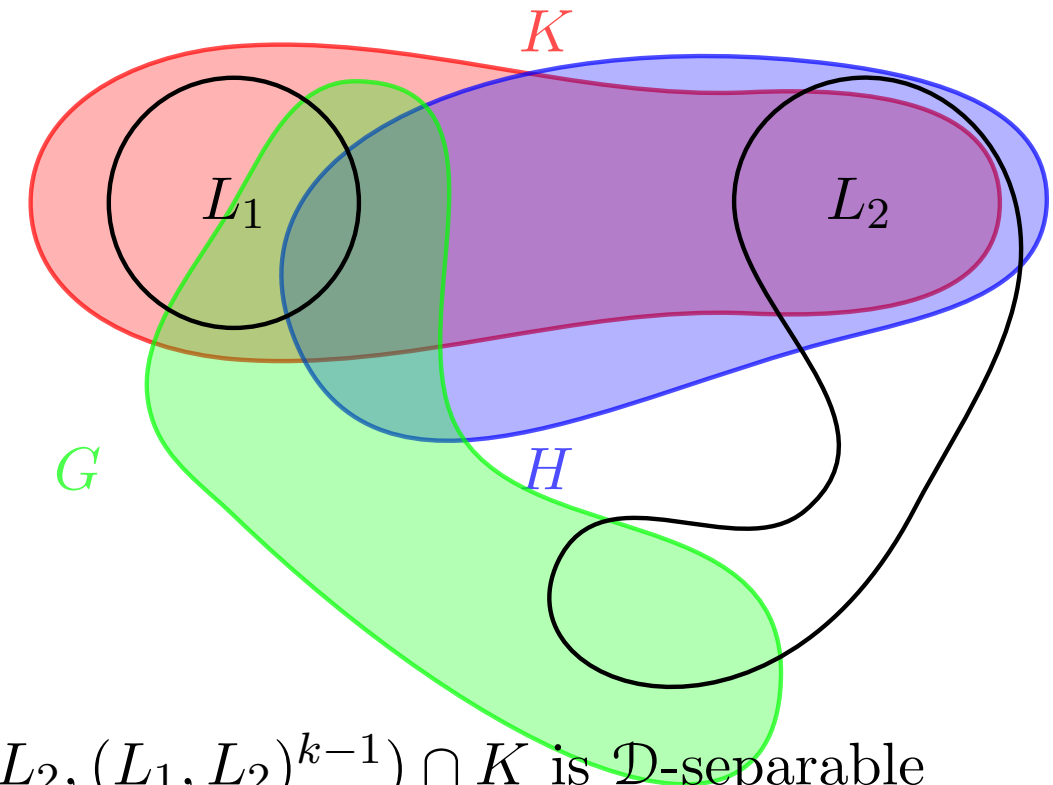

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$BPol(\mathcal{C})$-separation: Three main steps (2)

Getting rid of Boolean closure

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Step 1

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(K, L) $BPol(\mathcal{C})$-separable
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Solving generalized $Pol(\mathcal{C})$-separation:
Input: $(L_1, \ldots, L_n)$ (regular)
Output: Is $(L_1, \ldots, L_n)$ $Pol(\mathcal{C})$-separable ?

Generalized $Pol(\mathcal{C})$-separation: Approach (1)

Our input $(L_1, \ldots, L_n)$ is a **tuple of n regular languages**.

We do **not work directly with these languages**.

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One always looks at several inputs simultaneously.

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One always looks at several inputs simultaneously.

We use as **set of inputs** which has a special structure.

Generalized $Pol(\mathcal{C})$-separation: Approach (2)

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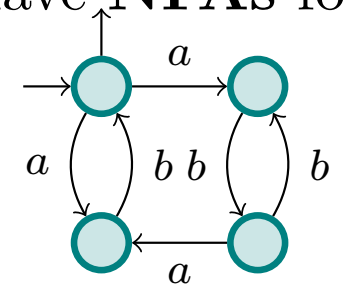
$\Rightarrow$ We have **NFAs** for these languages:



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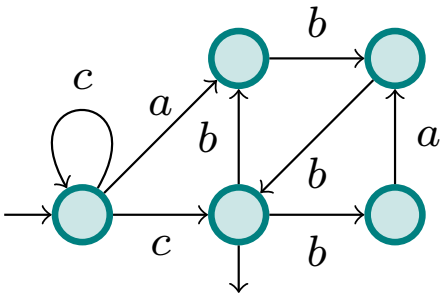

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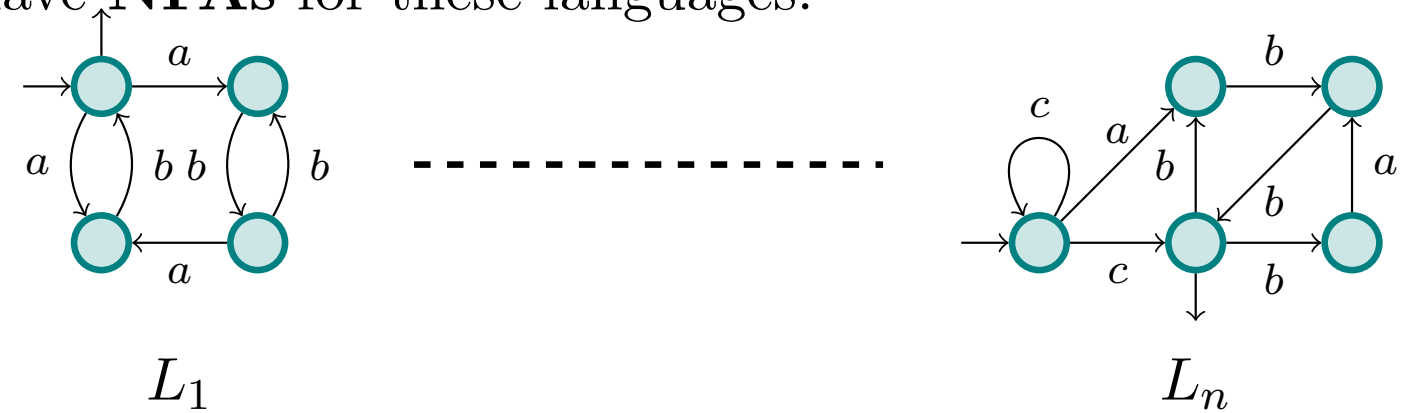

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We work with a **set of languages L** containing all languages

$$L_{p,q} = \{w \mid p \xrightarrow{w} q\} \quad (p, q \text{ two states of these } \mathbf{NFAs})$$

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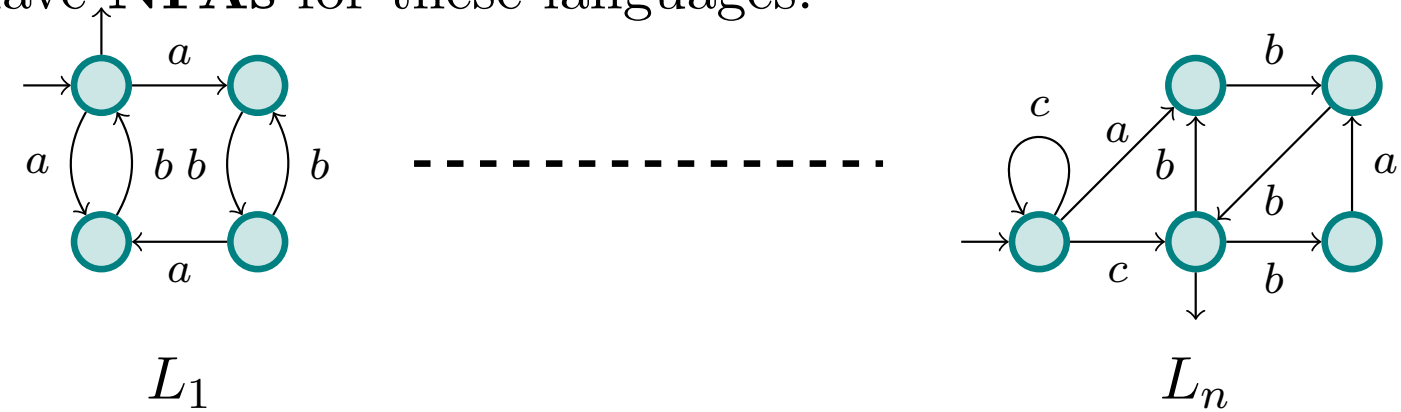

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Answer for $(L_1, \ldots, L_n)$ can then be extracted from this information.

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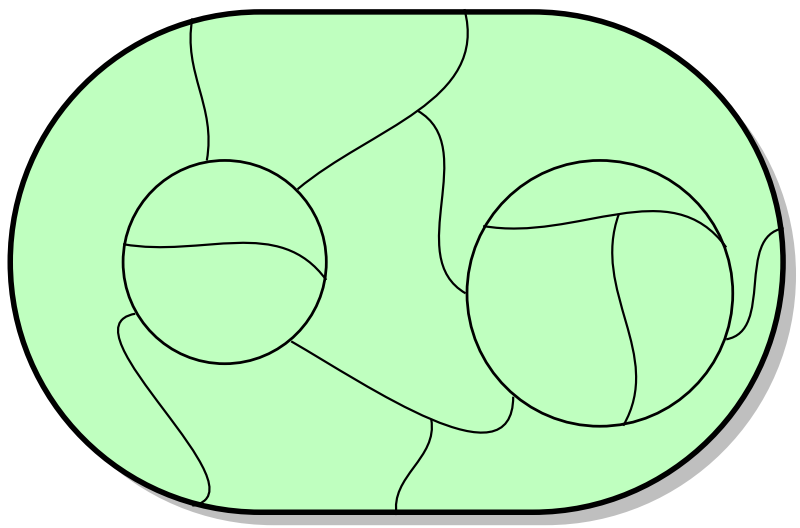

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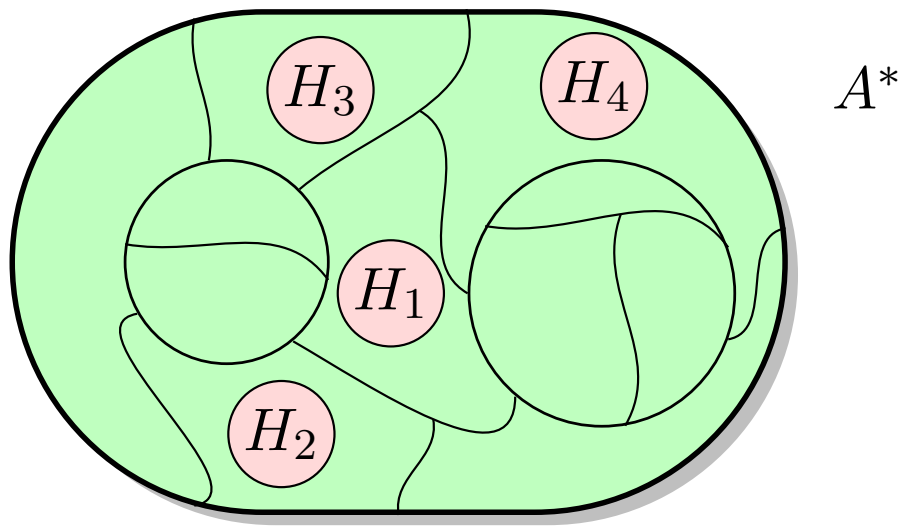

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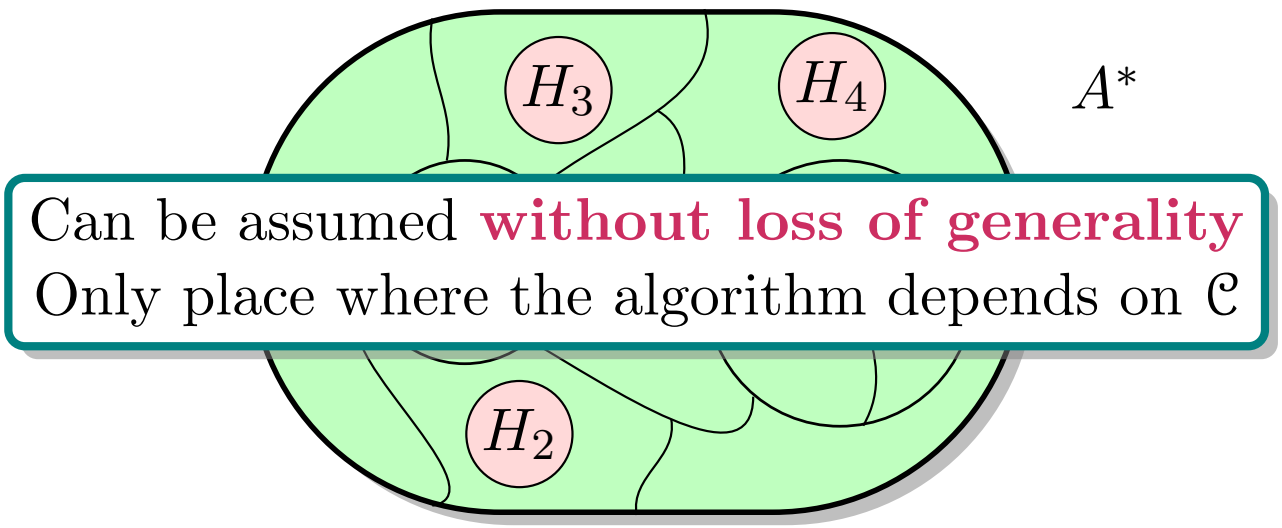

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Least fixpoint computation of $\mathcal{T}^n[\mathbf{L}]$

Given $n \geq 1$, we compute the set $\mathcal{T}^n[\mathbf{L}] \subseteq \mathbf{L}^n$ of all tuples $\overline{L} \in \mathbf{L}^n$ which are **not** $Pol(\mathcal{C})$**-separable**.

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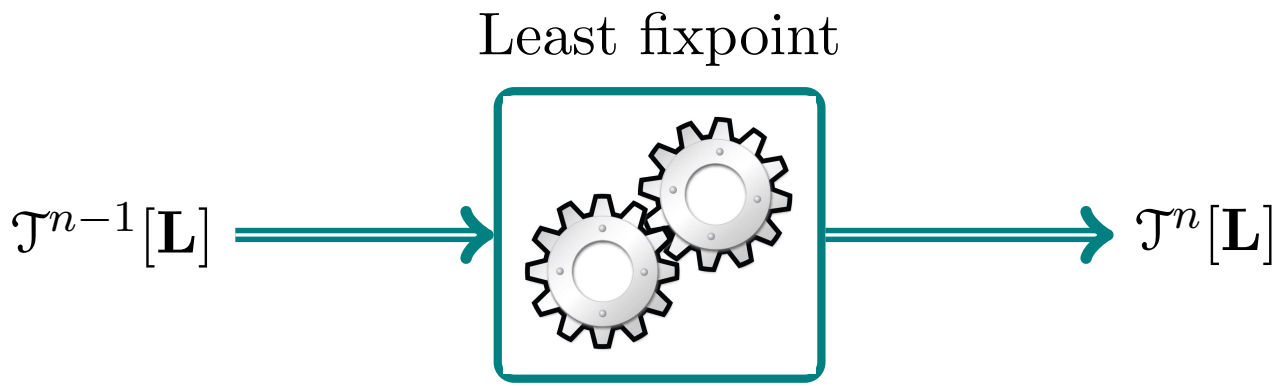

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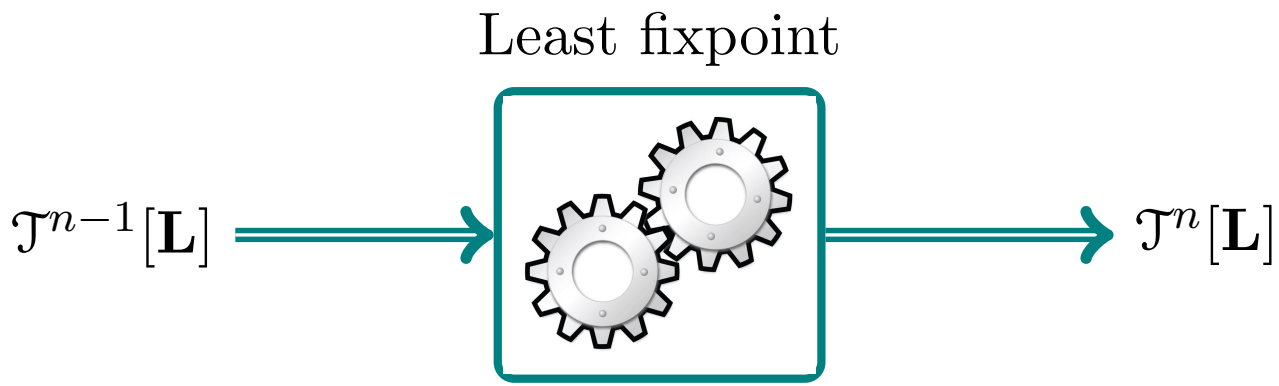

The procedure computes $\mathcal{T}^n[\mathbf{L}]$ from a subset of trivial elements and **adds more with operations until a fixpoint is reached**. Implementing these operations requires having $\mathcal{T}^{n-1}[\mathbf{L}]$ in hand.

$BPol(\mathcal{C})$-separation: Three main steps (3)

Step 1

Step 2

Step 3

Getting rid of Boolean closure

(K, L) $BPol(\mathcal{C})$-separable
iff
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Input: $(L_1, \ldots, L_n)$ (regular)
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Reuse Step 2 as a subprocedure
in our $BPol(\mathcal{C})$ algorithm

$BPol(\mathcal{C})$-separation - General approach

We continue to work with a set of languages $\mathbf{L}$ with appropriate properties.

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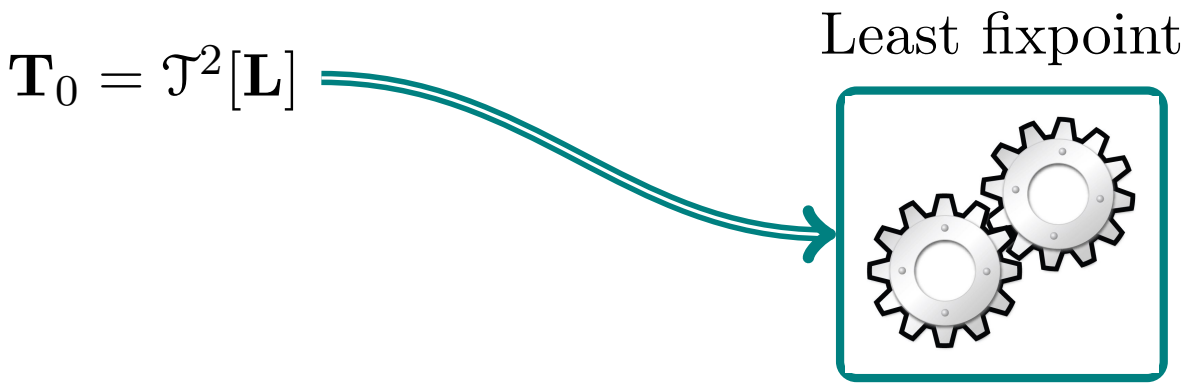

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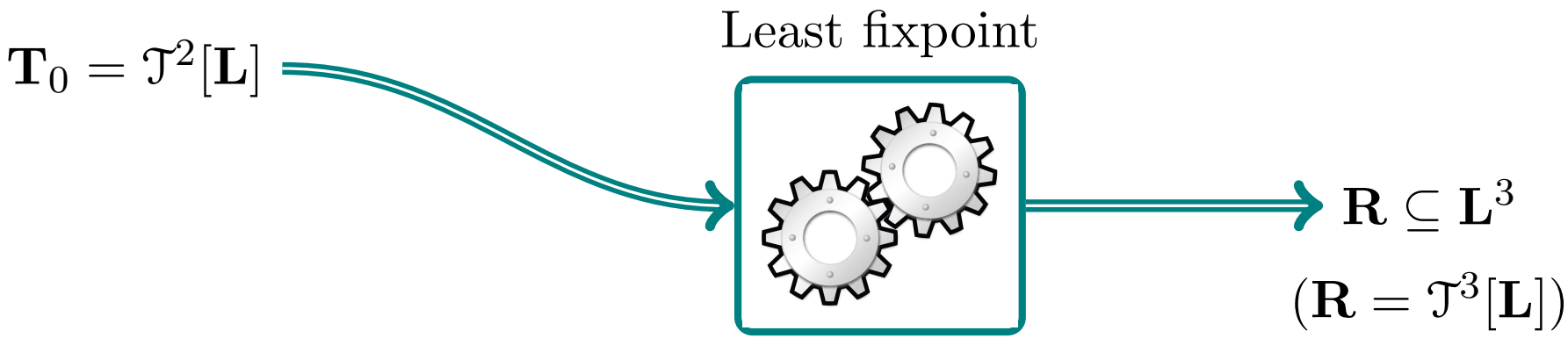

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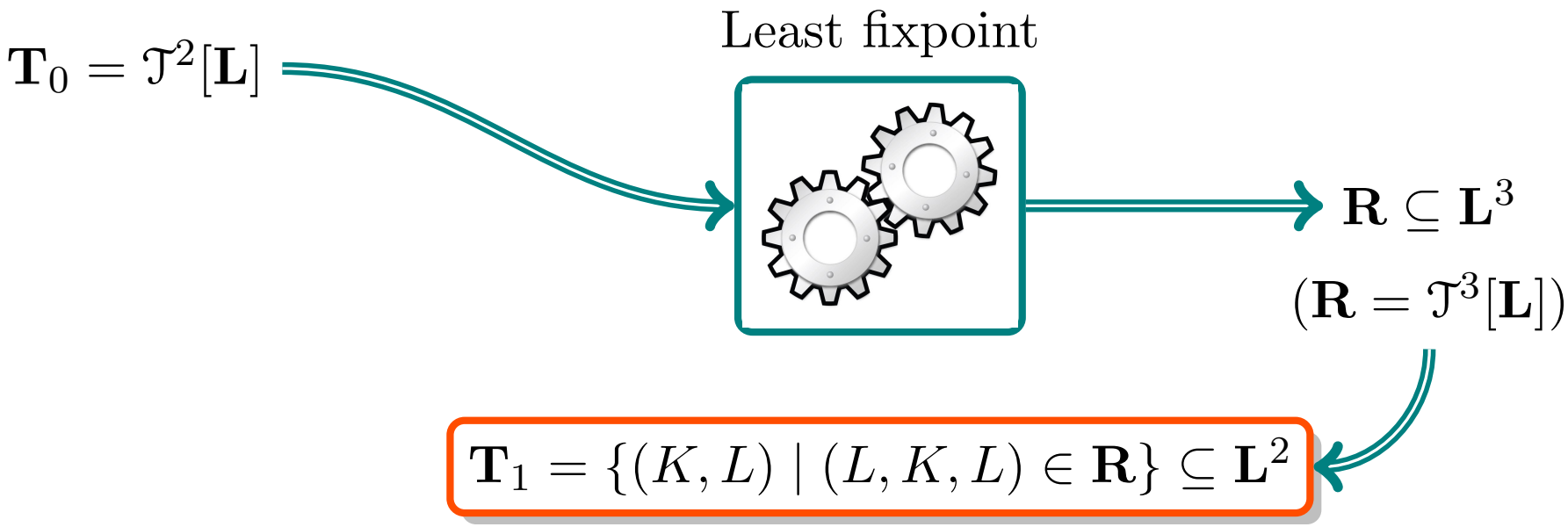

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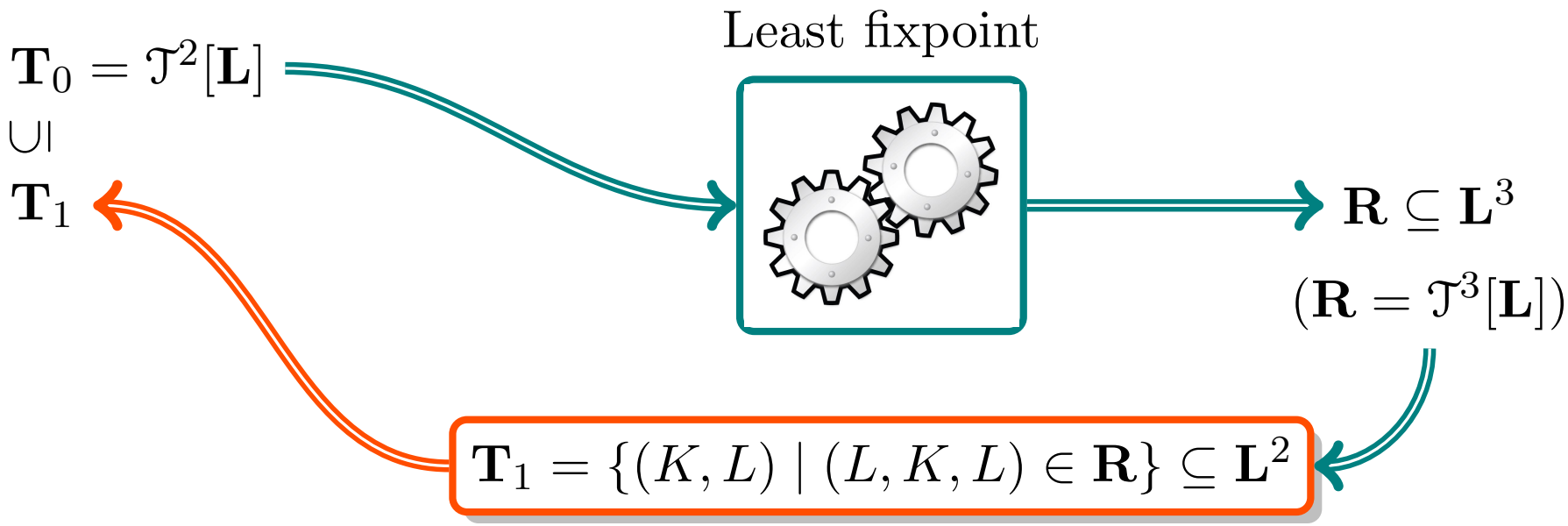

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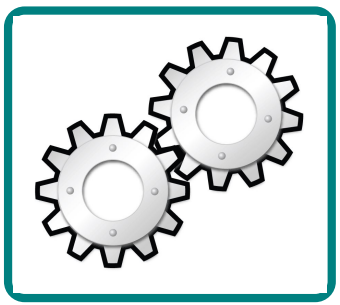

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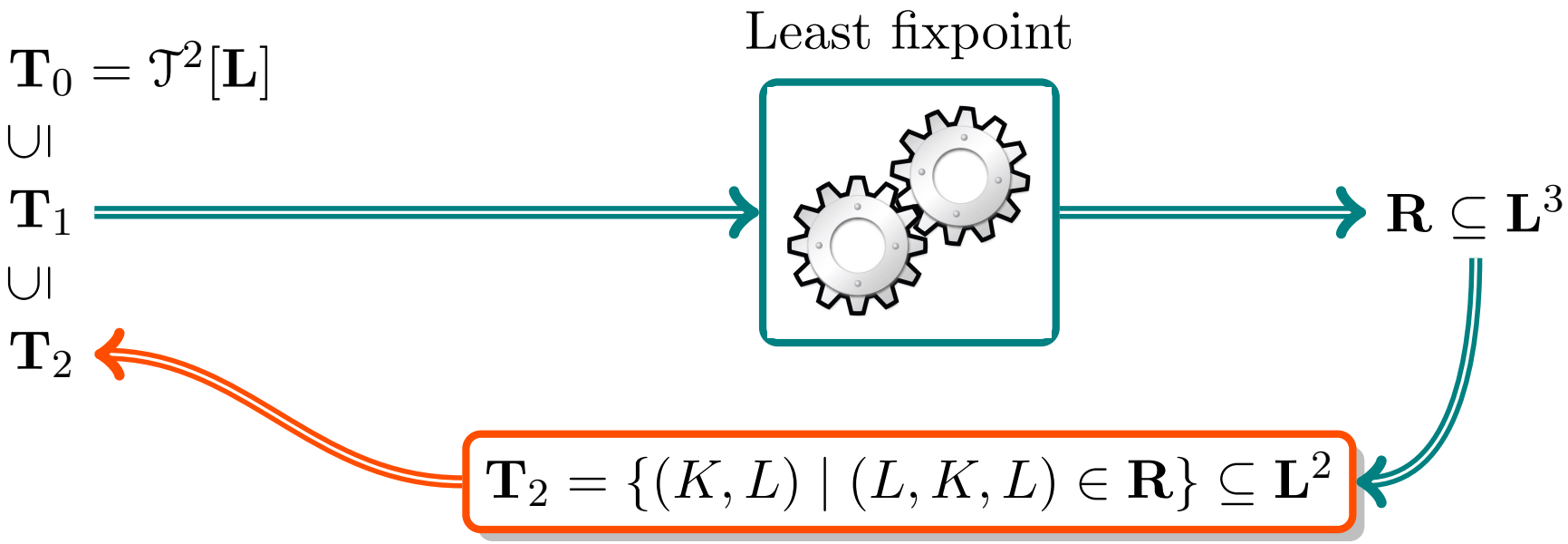

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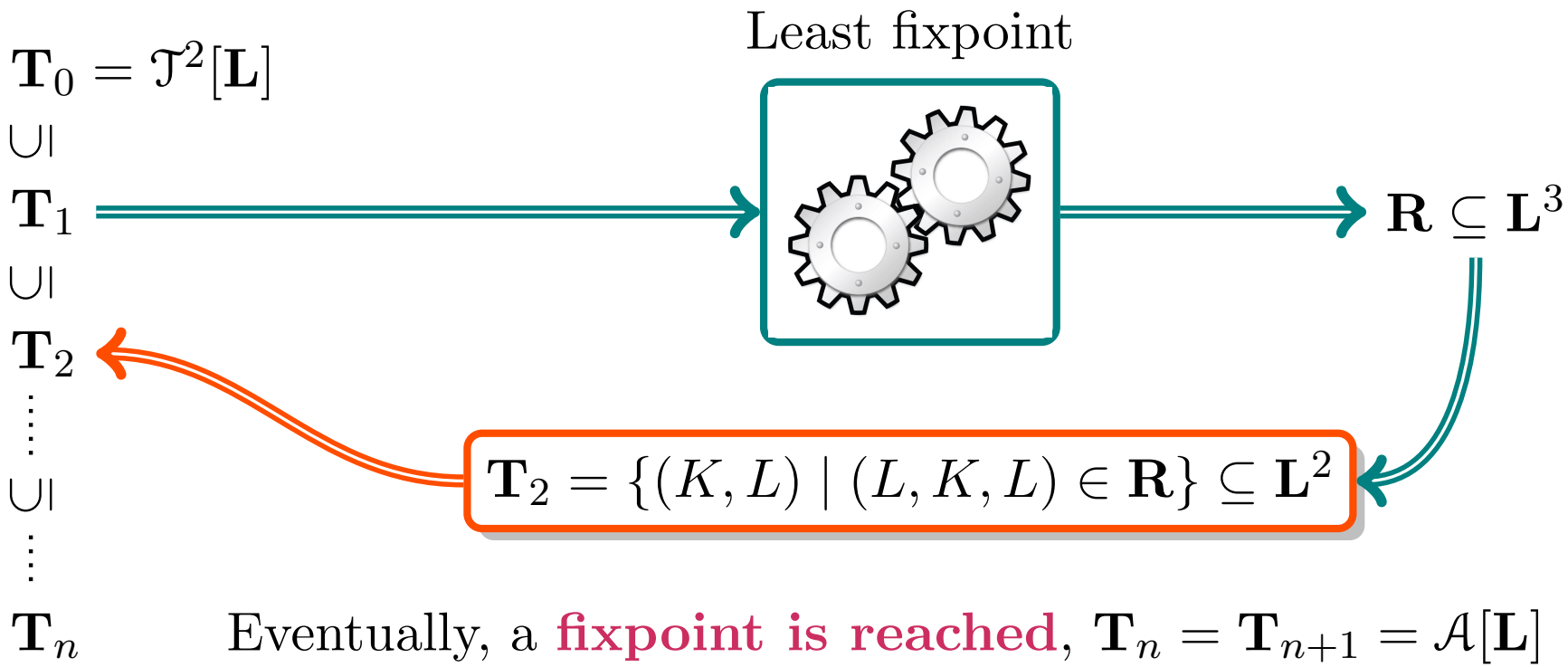

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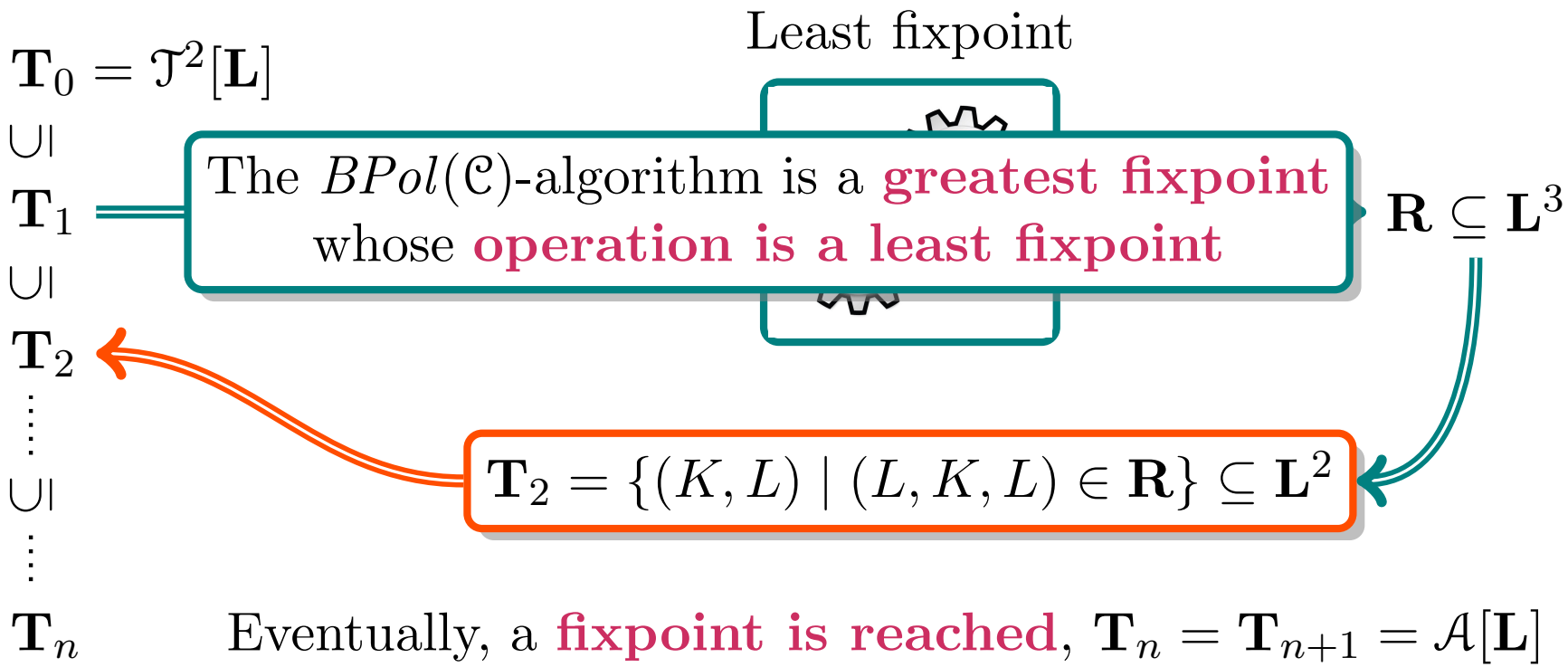

Conclusion

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Everything we know is captured by only **four generic results**:

1. $\mathcal{C}$ finite $\Rightarrow$ $Pol(\mathcal{C})$-separation decidable.
2. $\mathcal{C}$ finite $\Rightarrow$ $BPol(\mathcal{C})$-separation decidable.
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Future work

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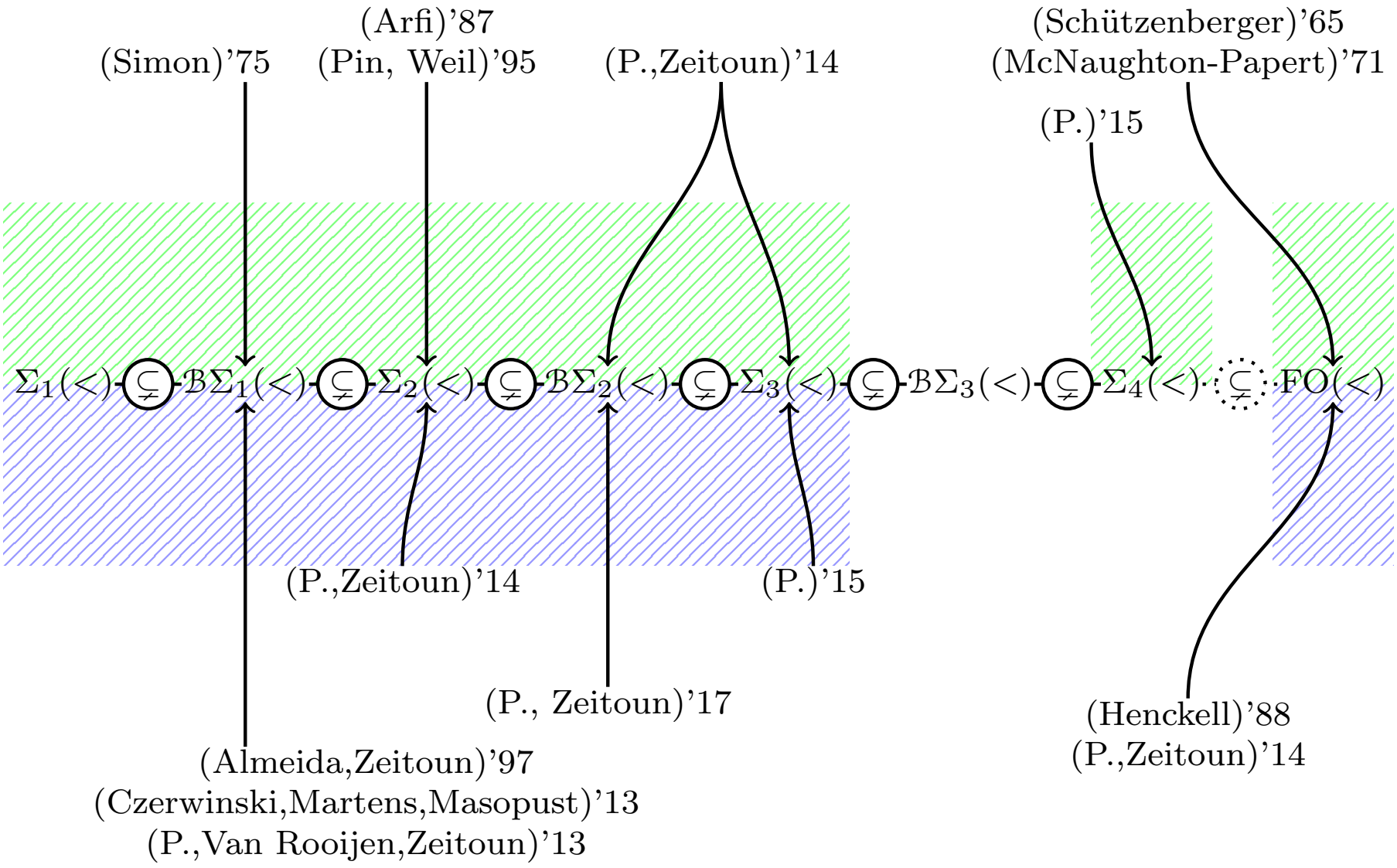
$\mathcal{B}\Sigma_2(<)$-"something" seems to be decidable which would yield a membership algorithm for $\mathcal{B}\Sigma_3(<)$.


(Simon)'75
(Arfi)'87
(Pin, Weil)'95
(P.,Zeitoun)'14
(Schützenberger)'65
(McNaughton-Papert)'71
(P.)'15
$\Sigma_1(<)$
$\mathcal{B}\Sigma_1(<)$
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$\mathcal{B}\Sigma_2(<)$
$\Sigma_3(<)$
$\mathcal{B}\Sigma_3(<)$
$\Sigma_4(<)$
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FO(<)
(P.,Zeitoun)'14
(P.)'15
(P., Zeitoun)'17
(Almeida,Zeitoun)'97
(Czerwinski,Martens,Masopust)'13
(P.,Van Rooijen,Zeitoun)'13
(Henckell)'88
(P.,Zeitoun)'14

Thank You