From star-free closure to mixed polynomial closure: the incredible story of operators.

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Classes of regular languages
and their *investigation*

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- Goal: investigate **sub-classes** of the regular languages.

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1. Classes based on **regular expressions**:
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2. Classes based on **logic**. By Büchi's theorem, MSO = REG:
   A restriction of MSO yields a sub-class.

The overused historical example: **first-order logic**

First-order logic over words (FO(<))

- Word: sequence of labeled positions that can be quantified:

$$a\ b\ b\ b\ c\ a\ a\ a \quad \in A^*$$
$$0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9$$

- Two kinds of predicates:
  1. for each letter $a \in A$, $a(x)$ selects positions x with label "a".
  2. single binary predicate for the (strict) order: $x < y$.
- A sentence defines a language:

$$\exists x \exists y\ a(x) \wedge b(y) \wedge x < y \wedge (\forall z\ x < z < y \Rightarrow c(z))$$
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Informal objective

"Understand" the **expressive power** of FO(<):

- What regular languages can we express?
- What are those that we cannot express ?

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- closed under **marked concatenation**:

$$\text{for a letter } a \in A \qquad K, L \mapsto KaL$$

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What does this mean ? Why does this give a membership algorithm ?

Monoids, regular languages: the (very) quick reminder

The basics

- A^* is a **monoid** (multiplication is word concatenation).
- We may look at **morphisms** $\alpha : A^* \to M$ into a monoid M.
- A language L is **recognized** by α if $L = \alpha^{-1}(F)$ for $F \subseteq M$.

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For each regular language L, there exists a **canonical morphism into a finite monoid** $\alpha_L : A^* \to M_L$ recognizing L:

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The syntactic morphism **can be computed**.
(monoid counterpart of the minimal automaton).

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"ω" is a computable number associated to a **finite** monoid M:
for every $t \in M$, t^ω is **idempotent** ($t^\omega = t^\omega t^\omega$).

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Important: constructive proof for 2) $\Rightarrow$ 1)

Normal forms for descriptions of star-free/first-order languages.

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In the talk, we look at variants allowing **at least the linear order**.

Defining families of logical classes:
Fragments of FO as ***operators***

Signature *associated to a class of languages* $\mathcal{C}$

We associate a **signature $\mathbb{S}_{\mathcal{C}}$ to every class $\mathcal{C}$**. It contains the labels and, for each $L \in \mathcal{C}$, a binary predicate $F_L(x, y)$ such that,

$$a_1 a_2 a_3 \cdots a_n \models F_L(i, j) \iff i < j \text{ and } a_{i+1} \cdots a_{j-1} \in L.$$

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Example: the trivial class $\mathcal{C} = \{\emptyset, A^*\}$

- $F_{A^*}(x, y)$ is equivalent to $x < y$.
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- $\Rightarrow$ we get $\mathcal{F}(\mathbb{S}_{\{\emptyset, A^*\}}) = \mathcal{F}(<)$.

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Definition robust when $\mathcal{C}$ is a **Boolean algebra** (union, intersection and complement) closed under **quotients**.

Key input classes: **Group languages** (1)

- Recognized by a **morphism into a finite group**.
- Equivalently recognized by a **permutation automaton**.

Permutation Automaton

Complete deterministic finite automaton (DFA) such that:

each letter induces a **permutation of the states**

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Example 1: even length ($A = \{a, b\}$)

$$
\begin{aligned}
a &: 1 \mapsto 2 \\
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b &: 1 \mapsto 2 \\
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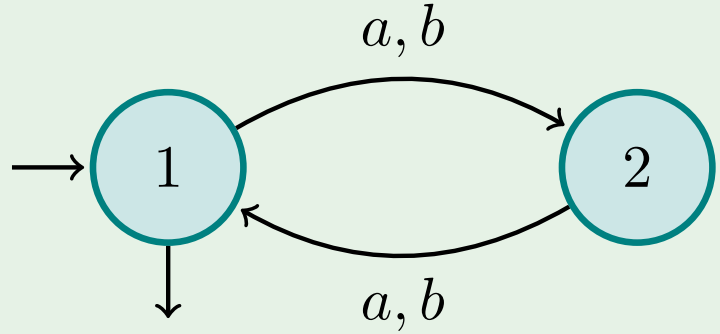
\end{aligned}
$$



Represents the class MOD: **modulo languages**.

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Example 2: even number of "b", abelian language ($A = \{a, b\}$)

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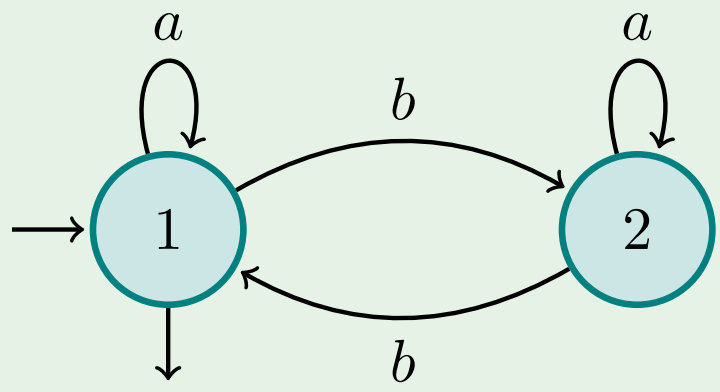

Represents the class AMT: **alphabet modulo testable languages**.

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Example 3: $(ab^*a + ba^*b)^*$ $(A = \{a, b\})$

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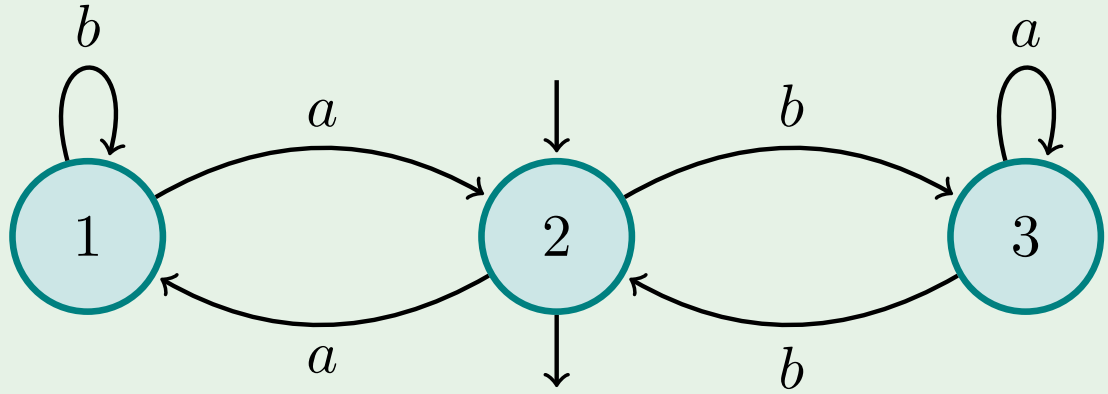

Represents the class GR: **all group languages**.

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For each $L \in \mathcal{G}$, $\mathbb{P}_{\mathcal{G}}$ contains unary predicate $P_L(x)$ such that,

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For each fragment $\mathcal{F}$, we look at **two main questions**:

1. Reformulating $\mathcal{C} \mapsto \mathcal{F}(\mathbb{S}_{\mathcal{C}})$ using "regular expressions".
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New objective

For a fragment of first-order logic $\mathcal{F}$ (e.g. FO itself):

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In the talk, we look at several prominent fragments.

Capturing first-order logic:
Star-free closure

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For an alphabet A, $\mathrm{SF}(\mathcal{C})$ is the least class of languages such that:

- contains $\mathcal{C}$.
- closed under **union** and **complement**.

$$K, L \mapsto K \cup L \qquad\qquad K \mapsto A^* \setminus K$$

- closed under **marked concatenation**:

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For each class $\mathcal{C}$, we have the correspondence $\mathrm{SF}(\mathcal{C}) = \mathrm{FO}(\mathbb{S}_{\mathcal{C}})$.

Membership: the **general case** (P., Zeitoun'19)

$\mathcal{C}$ a class, L a regular language, $\alpha : A^* \to M$ its syntactic morphism. The following properties are equivalent.

1. $L \in \mathrm{SF}(\mathcal{C}) = \mathrm{FO}(\mathbb{S}_{\mathcal{C}})$.
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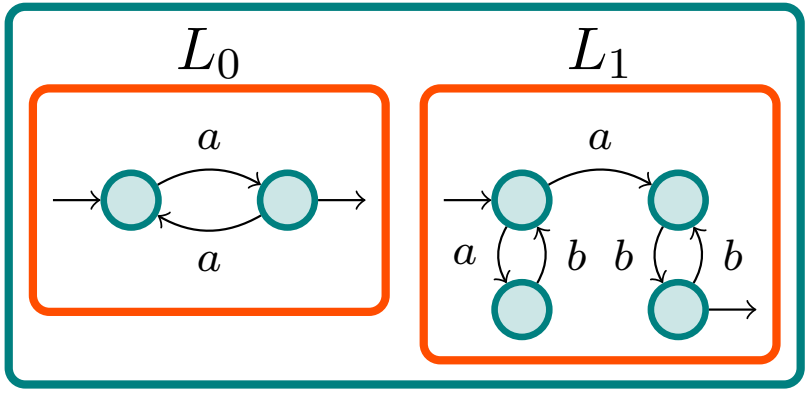
What is this weird condition ?!
Fortunately, the special case of **group** classes is simpler.

Towards the group case: **separation** and **pairs**

Separation problem for a class $\mathcal{C}$

INPUT: **two** regular languages



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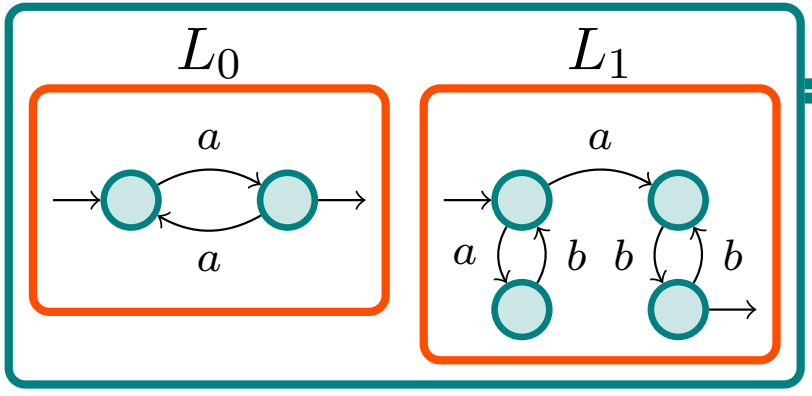

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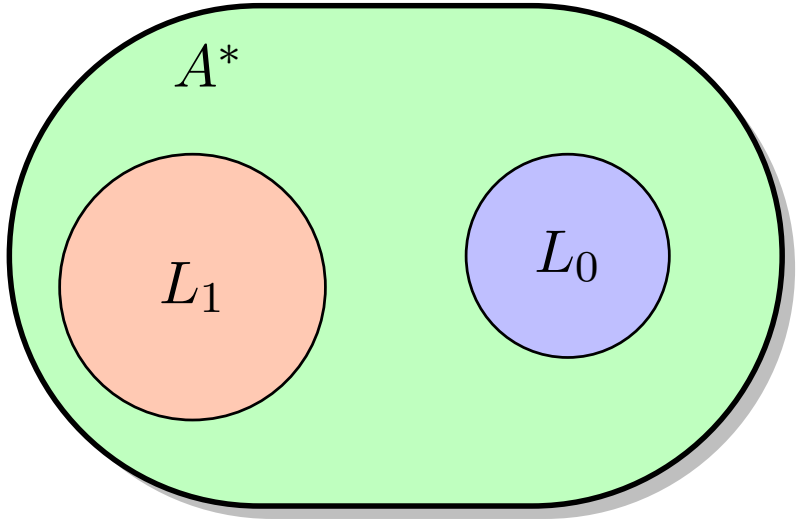

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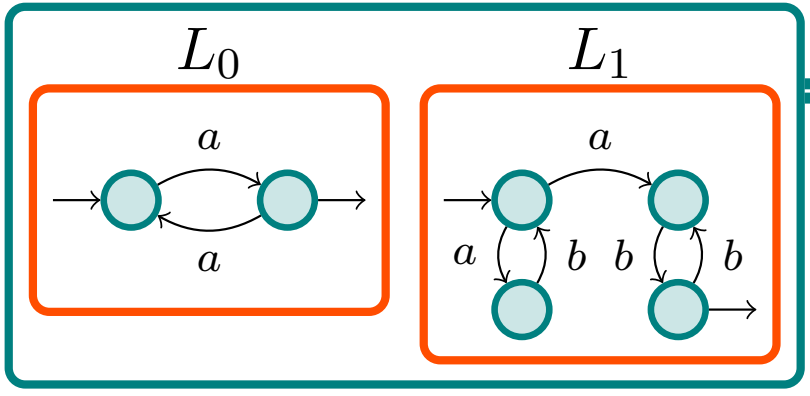

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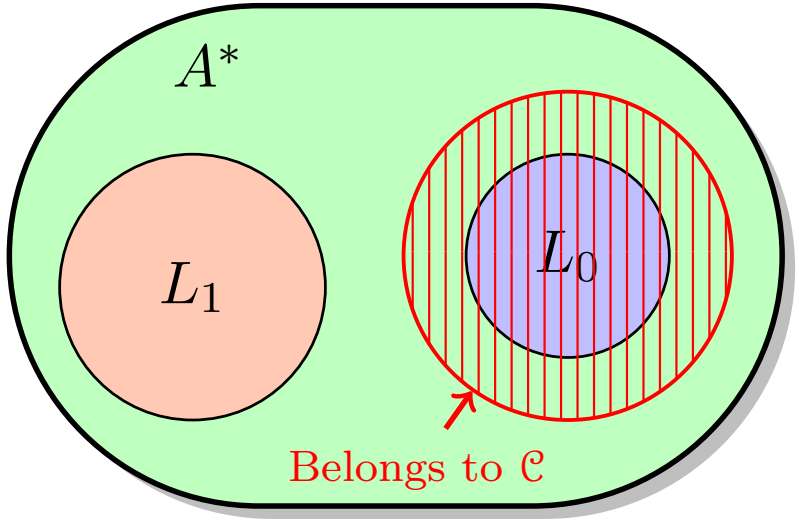

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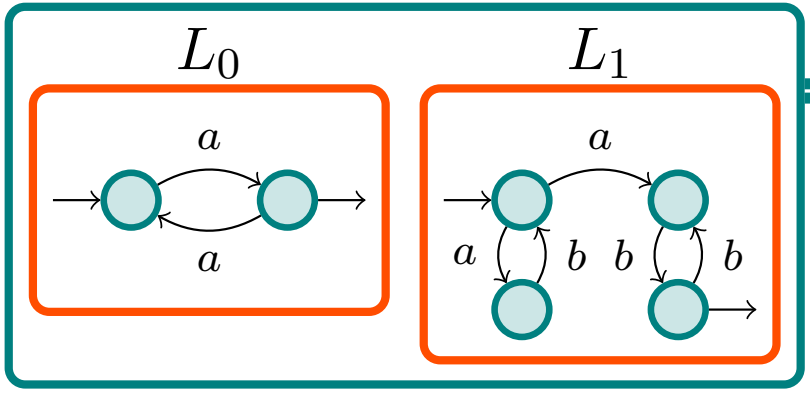

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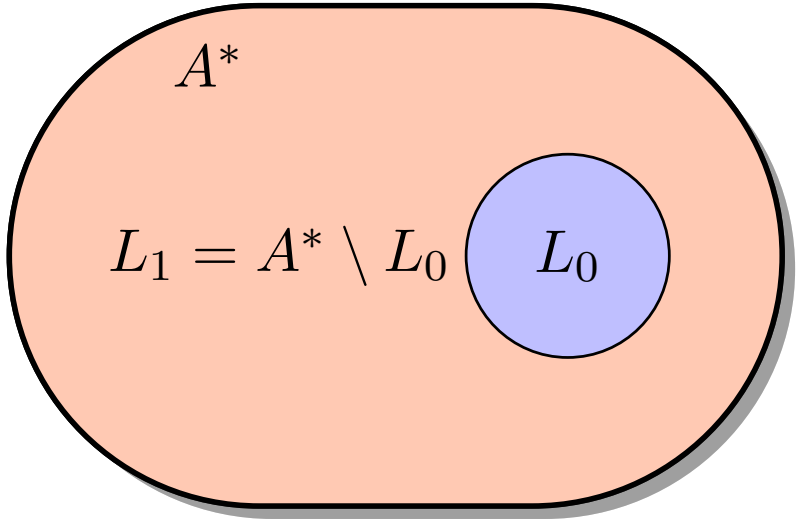

Effective **reduction** from membership to separation

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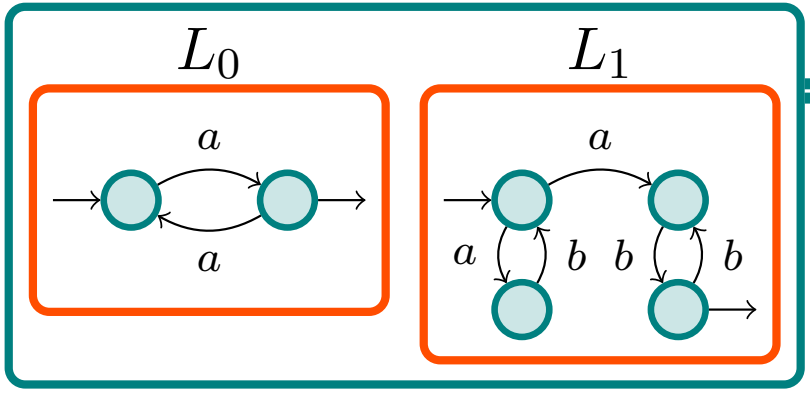

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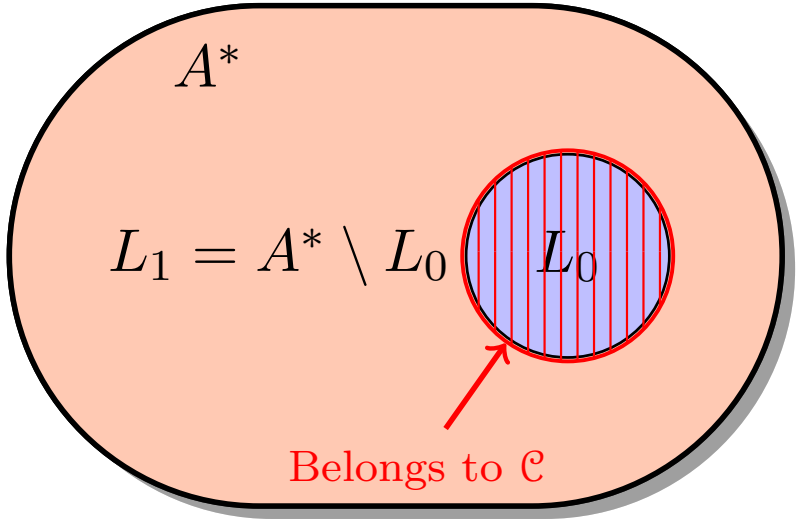

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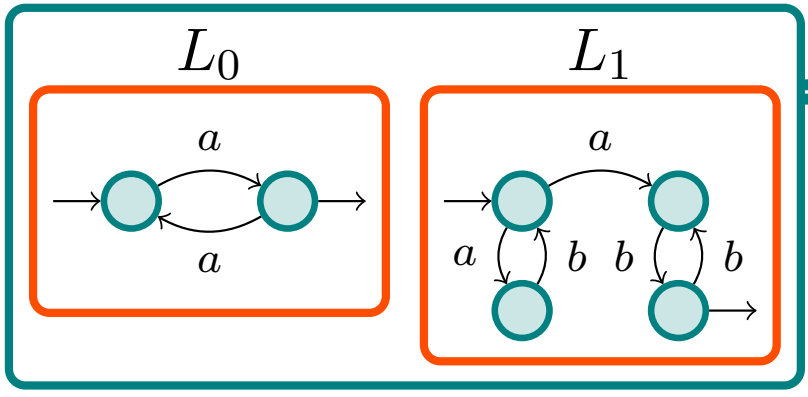

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The $\mathcal{C}$-pairs associated to a morphism

Given a morphism $\alpha : A^* \to M$ a **$\mathcal{C}$-pair for α** is a pair $(s, t) \in M^2$ such that $\alpha^{-1}(s)$ is **not** $\mathcal{C}$-separable from $\alpha^{-1}(t)$.

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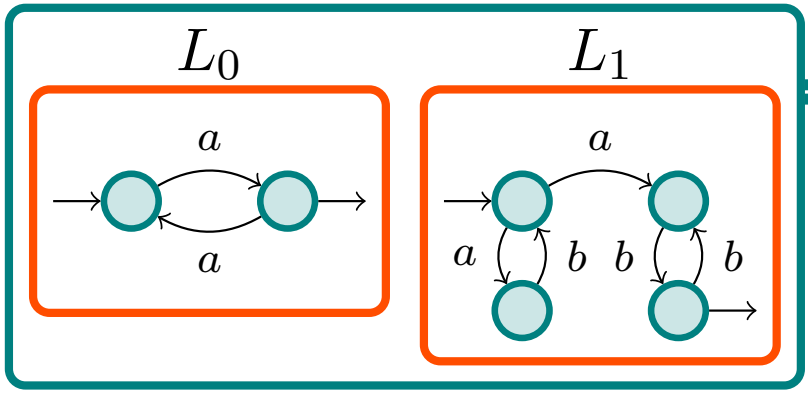

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Key point: $\mathcal{C}$-pairs can be computed when $\mathcal{C}$-separation is decidable.

Membership: the **group case** ($\mathcal{G}$ is a group class)

L a regular language and $\alpha : A^* \to M$ its syntactic morphism.
The following properties are equivalent.

1. $L \in \mathrm{SF}(\mathcal{G}) = \mathrm{FO}(<, \mathbb{P}_{\mathcal{G}})$.
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The story so far for a **group class** $\mathcal{G}$

Logic	Operator
first-order logic: $\mathrm{FO}(<, \mathbb{P}_{\mathcal{G}})$	star-free closure: $\mathrm{SF}(\mathcal{G})$

Membership decidable if **$\mathcal{G}$-separation decidable**.

Let's move to other operators:

Concatenation hierarchies

Concatenation hierarchies (Brzozowski,Cohen'71)

What is a "simple" star-free language?

- $A^*aA^*bcbA^*aA^*$.
- $(ab)^* = \overline{aA^* \cup A^*aaA^* \cup A^*bbA^* \cup A^*a}$.

What is a "complicated" star-free language

- $A^*a\left(\overline{A^*ab\left(\overline{A^*a\left(\overline{A^*bbA^*}\right)acA^*}\right)cc\left(\overline{A^*bcaA^*}\right)baA^*}\right)cbA^*$.

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Complicated = Alternation between concatenation and complement.

Intuition is mathematically validated:

- Deciding emptiness of a star-free language is **non-elementary**. (Meyer,Stockmeyer 73).
- Lower bound tied to alternations concatenation/complement.

Concatenation hierarchies (Brzozowski,Cohen'71)

For an input class $\mathcal{C}$ (**basis**), classify $\mathrm{SF}(\mathcal{C})$ into **half** and **full levels**.

Complexity measure: **alternation complement/concatenation**.

$$0 \longrightarrow \tfrac{1}{2} \longrightarrow 1 \longrightarrow \tfrac{3}{2} \longrightarrow 2 \longrightarrow \tfrac{5}{2} \longrightarrow 3 \cdots\cdots \mathrm{SF}(\mathcal{C})$$

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Polynomial closure

$Pol(\mathcal{D})$ finite unions of **marked products** $L_0 a_1 L_1 \cdots a_n L_n$ where $a_1, \ldots, a_n \in A$ and $L_0, \ldots, L_n \in \mathcal{D}$.

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Boolean closure

$\mathit{Bool}(\mathcal{D})$: least **Boolean algebra** containing $\mathcal{D}$ (closure under union and complement).

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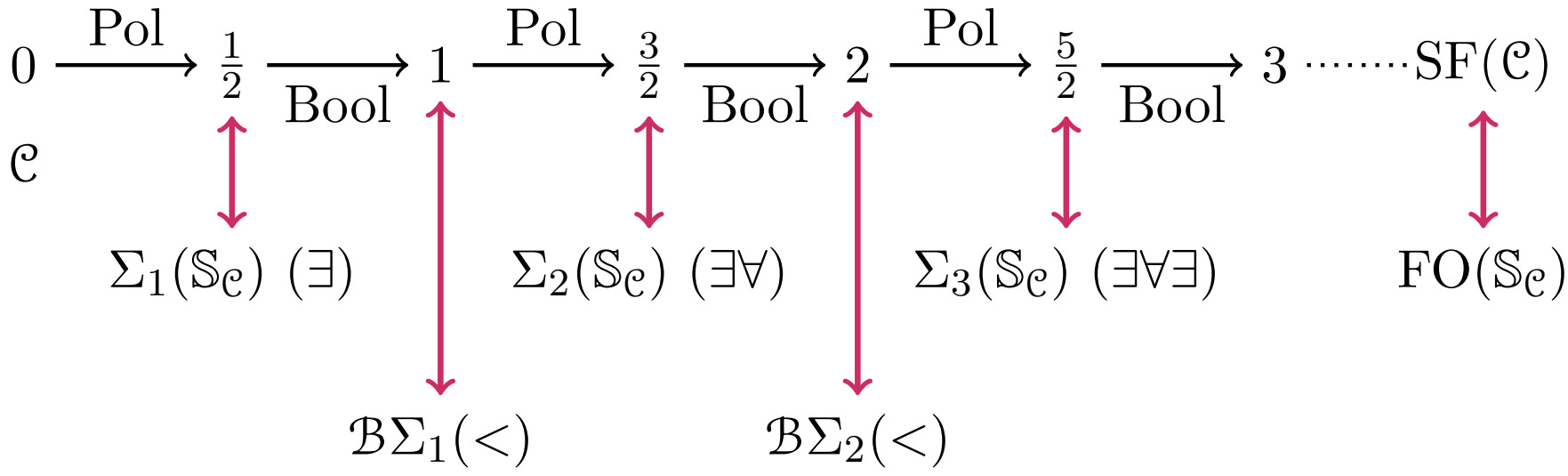
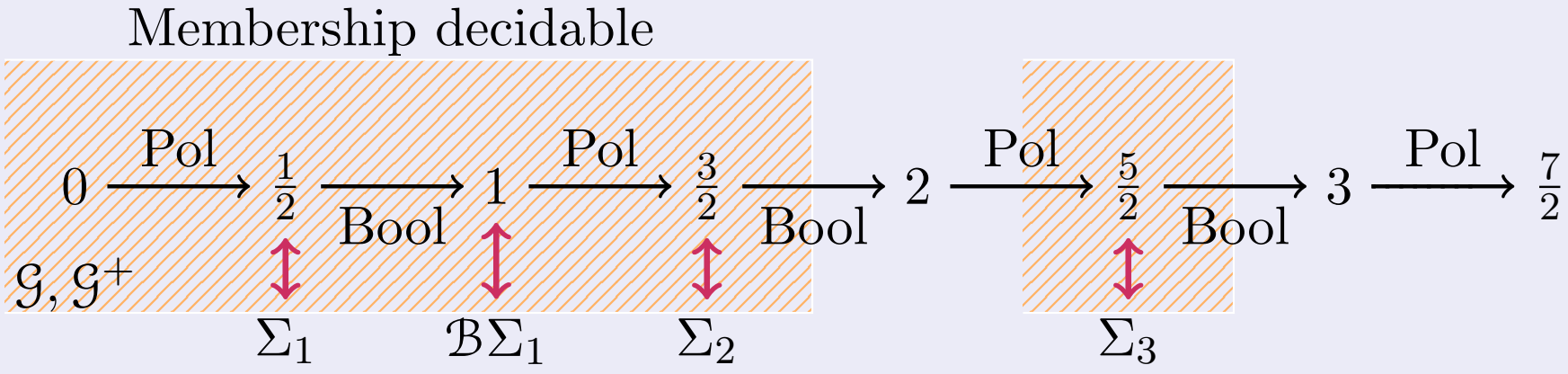


Logical point of view: **quantifier alternation** (Thomas 1982).

Overview of the membership results for group classes

$\mathcal{G}$ a group class **with decidable separation**. The bases $\mathcal{G}$ and $\mathcal{G}^+$:



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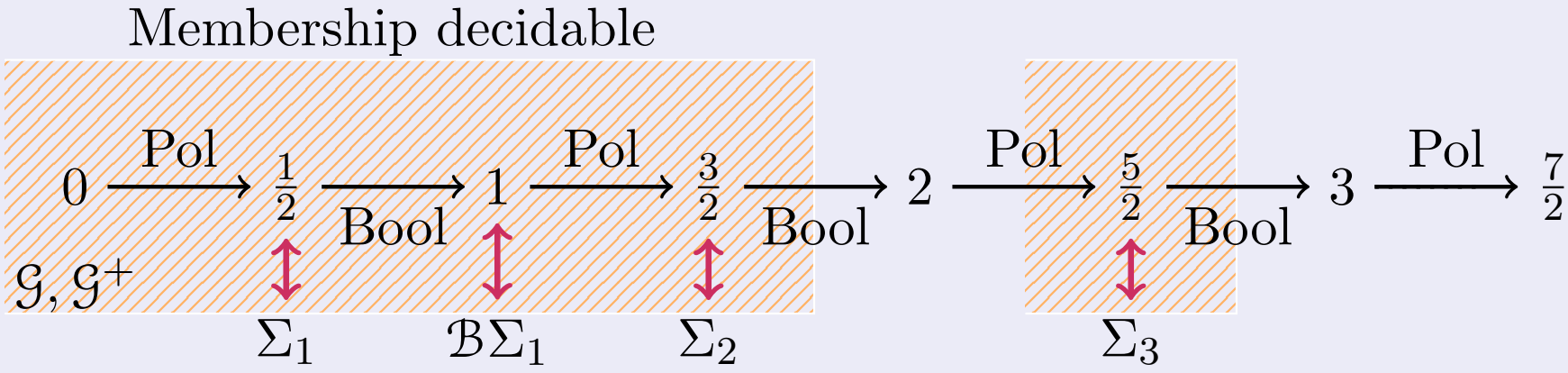
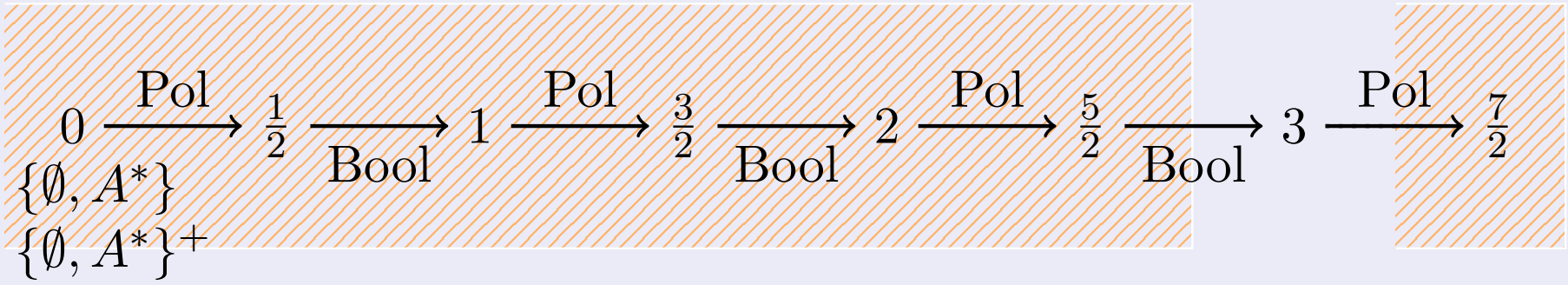


Specialized techniques for the bases $\{\emptyset, A^*\}$ and $\{\emptyset, A^*\}^+$:



In other words, we may also handle the classes:

- $\mathcal{B}\Sigma_2(<)$, $\Sigma_4(<)$, $\mathcal{B}\Sigma_2(<, +1)$ and $\Sigma_4(<, +1)$.

Membership for the **level one** ($\mathcal{G}$ is a group class)

Let us illustrate with the class $Bool(Pol(\mathcal{G})) = \mathcal{B}\Sigma_1(<, \mathbb{P}_\mathcal{G})$.
We write it $BPol(\mathcal{G})$ for short.

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A similar characterization holds for $BPol(\mathcal{G}^+) = \mathcal{B}\Sigma_1(<, +1, \mathbb{P}_\mathcal{G})$.

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$BPol(\mathcal{G})$-**separation** is decidable when $\mathcal{G}$-separation is decidable.
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$\mathcal{B}\Sigma_1(<, \mathbb{P}_{\mathcal{G}})$ and $\mathcal{B}\Sigma_1(<, +1, \mathbb{P}_{\mathcal{G}})$	$BPol(\mathcal{G})$ and $BPol(\mathcal{G}^+)$

In **all cases**, membership decidable if **$\mathcal{G}$-separation decidable**.

A weaker operator:
Unambiguous polynomial closure

Unambiguous polynomial closure: definition

$L_0 a_1 L_1 \cdots a_n L_n$ is **unambiguous** if every word $w \in L_0 a_1 L_1 \cdots a_n L_n$ admits a **unique** decomposition witnessing this membership:

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$$\text{if } u_i, v_i \in L_i \text{ for all } i \leq n \text{ and } u_0 a_1 u_1 \cdots a_n u_n = v_0 a_1 v_1 \cdots a_n v_n,$$
$$\text{then } u_i = v_i \text{ for every } i \leq n.$$

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Tied to the **product itself**, not only on the resulting language

- $b^*a(a+b)^*$ is unambiguous.
- $(a+b)^*a(a+b)^*$ is **not** unambiguous.

These two products evaluate to the same language.

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Unambiguous polynomial closure

$UPol(\mathcal{D})$: finite **disjoint** unions of **unambiguous** marked products $L_0 a_1 L_1 \cdots a_n L_n$ where $a_1, \ldots, a_n \in A$ and $L_0, \ldots, L_n \in \mathcal{D}$.

$UPol$ defined since the 70s/80s (Schützenberger'76, Pin'81).

Membership for UPol (P., Zeitoun'18)

$\mathcal{C}$ a class, L a regular language, $\alpha : A^* \to M$ its syntactic morphism. The following properties are equivalent:

1. $L \in UPol(\mathcal{C})$.
2. for all $s, t \in M$, if (s, t) is a $\mathcal{C}$-pair, then $s^{\omega+1} = s^\omega t s^\omega$.

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We can do **better**.

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$s \sim_{\mathcal{C}} t$ if and only if $s \in F \Leftrightarrow t \in F$ for all $F \subseteq M$ s.t. $\alpha^{-1}(F) \in \mathcal{C}$.

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i.e., $UPol$ **preserves the decidability of membership**.

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Using **group classes** directly is **pointless**:

For every group class $\mathcal{G}$, we have $UPol(\mathcal{G}) = \mathcal{G}$.

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The **important input classes** are $BPol(\mathcal{G})$ and $BPol(\mathcal{G}^+)$.

Logical characterization of UPol: **two-variable FO**

FO^2: the **two-variable** restriction of FO

At most two (reusable) variables are allowed in an FO^2 sentence:

$$\exists x \exists y \left(a(x) \wedge b(y) \wedge x < y \wedge (\exists x \left(c(x) \wedge y < x\right))\right).$$

This FO^2 sentence defines $A^* a A^* b A^* c A^*$.

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For every group class $\mathcal{G}$, the following correspondences hold:

- $\mathrm{FO}^2(<, \mathbb{P}_\mathcal{G}) = \mathit{UPol}(\mathit{BPol}(\mathcal{G}))$.
- $\mathrm{FO}^2(<, +1, \mathbb{P}_\mathcal{G}) = \mathit{UPol}(\mathit{BPol}(\mathcal{G}^+))$.

(This was first proved by Thérien and Wilke'98 for $\mathcal{G} = \{\emptyset, A^*\}$).

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Consequence: if $\mathcal{G}$-**separation** is decidable, membership is decidable for $\mathrm{FO}^2(<, \mathbb{P}_\mathcal{G})$ and $\mathrm{FO}^2(<, +1, \mathbb{P}_\mathcal{G})$.

The story so far for a **group class** $\mathcal{G}$

Logic	Operator
first-order logic: $\mathrm{FO}(<, \mathbb{P}_{\mathcal{G}})$	star-free closure: $\mathrm{SF}(\mathcal{G})$
$\mathcal{B}\Sigma_1(<, \mathbb{P}_{\mathcal{G}})$ and $\mathcal{B}\Sigma_1(<, +1, \mathbb{P}_{\mathcal{G}})$	$BPol(\mathcal{G})$ and $BPol(\mathcal{G}^+)$
two-variable first-order logic: $\mathrm{FO}^2(<, \mathbb{P}_{\mathcal{G}})$ and $\mathrm{FO}^2(<, +1, \mathbb{P}_{\mathcal{G}})$	$UPol(BPol(\mathcal{G}))$ and $UPol(BPol(\mathcal{G}^+))$

In **all cases**, membership decidable if **$\mathcal{G}$-separation decidable**.

Let's look at even weaker operators:

Deterministic polynomial closures

Deterministic marked polynomial closures

Consider a marked product $L_0 a_1 L_1 \cdots a_n L_n$. For $1 \leq i \leq n$, we write $L'_i = L_0 a_1 L_1 \cdots a_{i-1} L_{i-1}$ and $L''_i = L_i a_{i+1} \cdots L_{n-1} a_n L_n$.

$L_0 a_1 L_1 \cdots a_n L_n$ is **left deterministic** if $L'_i \cap L'_i a_i A^* = \emptyset$ for all i.

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- $A^*\ a\ b^*$ is right deterministic.
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$L_0 a_1 L_1 \cdots a_n L_n$ is **mixed deterministic** if either $L'_i \cap L'_i a_i A^* = \emptyset$ or $L''_i \cap A^* a_i L''_i = \emptyset$ for all i.

- $a^*\ b\ A^*\ a\ b^*$ is mixed deterministic (but neither left nor right).
- $(ac)^*\ a\ (ba)^*$ is **not** mixed deterministic (but it is unambiguous).

Deterministic marked polynomial closures

Left, right and mixed polynomial closures

Three new operators $LPol(\mathcal{D})$, $RPol(\mathcal{D})$ and $MPol(\mathcal{D})$.

$MPol(\mathcal{D})$: finite **disjoint** unions of **mixed deterministic** marked products $L_0 a_1 L_1 \cdots a_n L_n$ where $L_0, \ldots, L_n \in \mathcal{D}$.

$LPol/RPol$ defined alongside $UPol$ (Schützenberger'76, Pin'81).
On the other hand, $MPol$ **is new** (P.'22).

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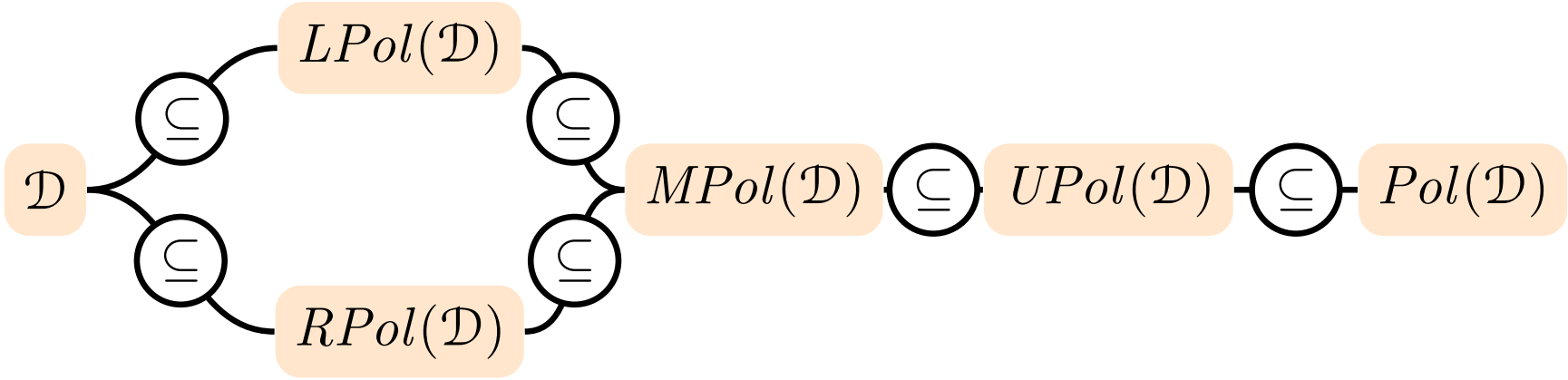

I will now convince you that $MPol$ is the **best operator ever**.

LPol, RPol, MPol: algebraic characterizations (P.'22)

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$\Rightarrow LPol, RPol, MPol$ **preserve the decidability of membership**.

Logical application
of **mixed polynomial closure.**

The quantifier alternation hierarchy of FO^2

For $n \in \mathbb{N}$, $\mathcal{B}\Sigma_n^2$ denotes the **formulas** which belong to both:

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I will use my own approach based on **mixed polynomial closure**.
Yet, there is a connection to the Kufleitner-Weil approach.

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Design principle: in general $LPol(\mathcal{C})$ and $RPol(\mathcal{C})$ are **incomparable**.

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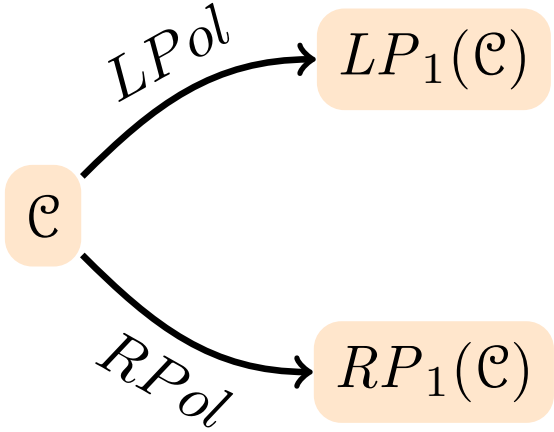

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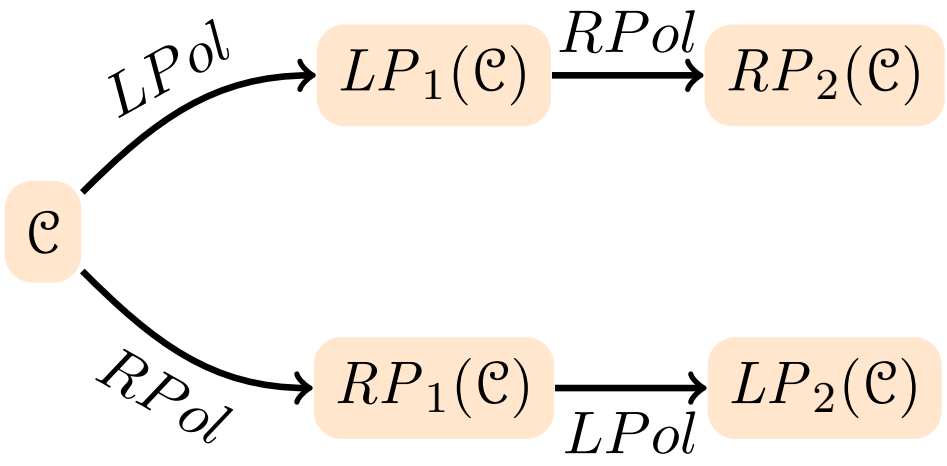

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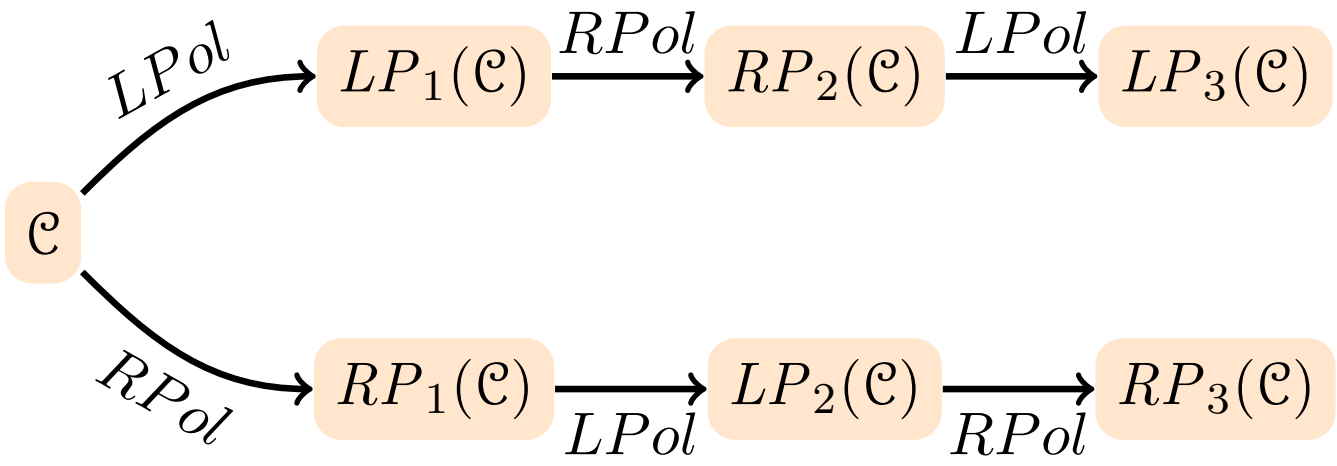

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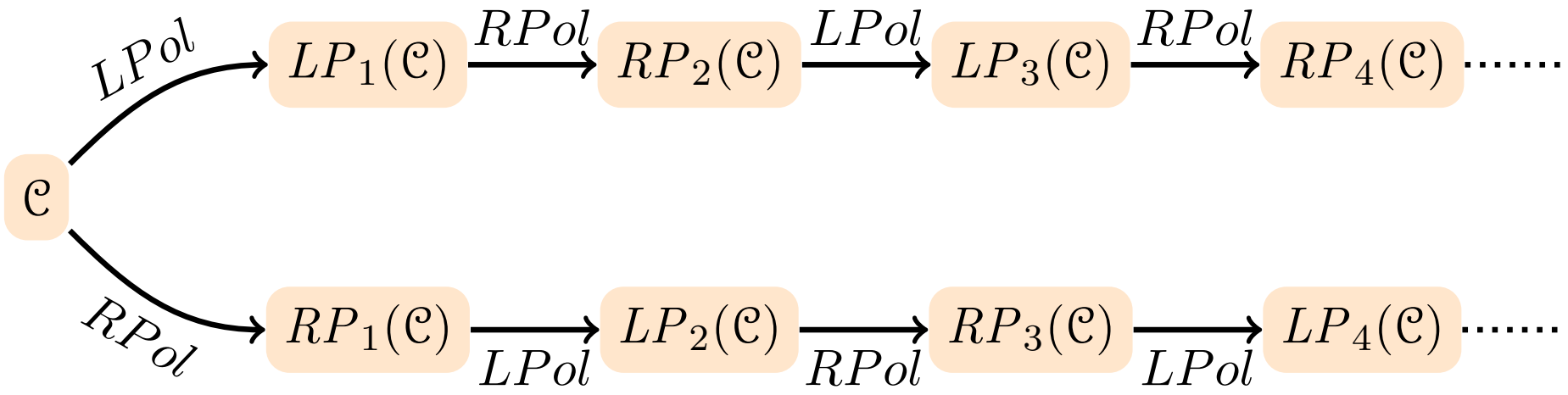

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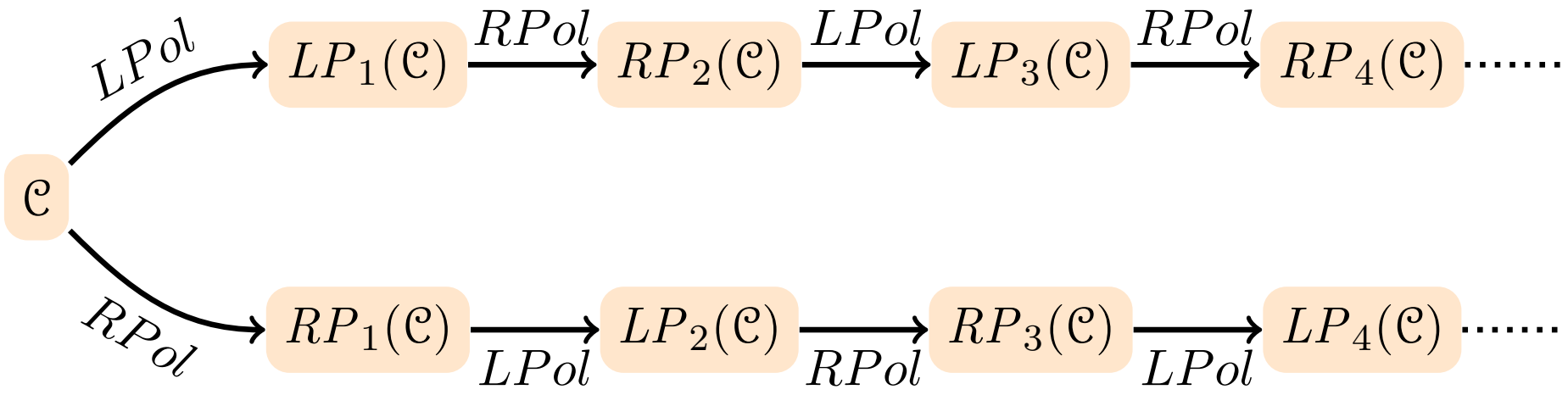

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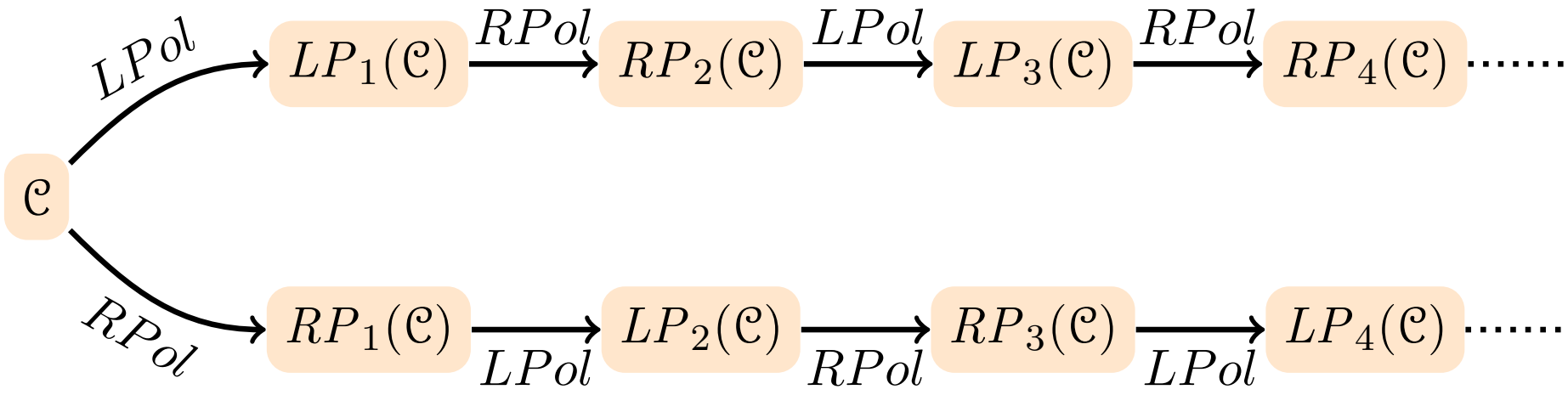

The **union of all levels** is $UPol(\mathcal{C})$ (P., Zeitoun'18).

If $\mathcal{C}$-membership is decidable, then the problem is also decidable for **all** levels in this hierarchy.

Deterministic hierarchies (2)

Consider the class $BPol(\{\emptyset, A^*\})$.

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This fails in general for the logics $\mathcal{B}\Sigma_1^2(<, \mathbb{P}_\mathcal{G})$ and $\mathcal{B}\Sigma_1^2(<, +1, \mathbb{P}_\mathcal{G})$.

Let us explain why.

The approach based on **mixed polynomial closure**

Dealing with the first level (applies to **all input classes**)

$\mathcal{C}$ a class. Then $\mathcal{B}\Sigma_1(\mathbb{S}_\mathcal{C}) = \mathcal{B}\Sigma_1^2(\mathbb{S}_\mathcal{C}) = BPol(\mathcal{C})$.

Hence, if $\mathcal{G}$ group class, then,

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Climbing with $MPol$ (specific to the **group input classes**)

$\mathcal{G}$ a group class. Then, for every $n \geq 1$,

- $\mathcal{B}\Sigma_{n+1}^2(<, \mathbb{P}_\mathcal{G}) = MPol(\mathcal{B}\Sigma_n^2(<, \mathbb{P}_\mathcal{G}))$.
- $\mathcal{B}\Sigma_{n+1}^2(<, +1, \mathbb{P}_\mathcal{G}) = MPol(\mathcal{B}\Sigma_n^2(<, +1, \mathbb{P}_\mathcal{G}))$.

$\Rightarrow$ If $\mathcal{G}$ has decidable separation, then membership is decidable for **all** levels $\mathcal{B}\Sigma_n^2(<, \mathbb{P}_\mathcal{G})$ and $\mathcal{B}\Sigma_n^2(<, +1, \mathbb{P}_\mathcal{G})$.

Connection with the Kufleitner-Weil characterization

First result: intersection levels can be climbed with $MPol$ (P.'22)

For every class $\mathcal{C}$ and $n \geq 1$,

$$LP_{n+1}(\mathcal{C}) \cap RP_{n+1}(\mathcal{C}) = MPol(LP_n(\mathcal{C}) \cap RP_n(\mathcal{C})).$$

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Second result: specific to $BPol(\{\emptyset, A^*\})$ (standard)

$$LP_1(BPol(\{\emptyset, A^*\})) \cap RP_1(BPol(\{\emptyset, A^*\})) = BPol(\{\emptyset, A^*\}).$$

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Combined with the "$MPol$ characterization" of the levels $\mathcal{B}\Sigma_n^2(<)$, this yields the Kufleitner-Weil characterization:

$$\mathcal{B}\Sigma_n^2(<) = LP_n(BPol(\{\emptyset, A^*\})) \cap RP_n(BPol(\{\emptyset, A^*\})) \quad \text{for all } n \geq 1.$$

To conclude : another fun result about $MPol$

- Reminder: $LP_n(\mathcal{C})$ and $RP_n(\mathcal{C})$ are incomparable in general.
- $\Rightarrow$ let's look at the least Boolean algebra containing both classes:

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Theorem: joined levels can be climbed with $MPol$ (P.'22)

For every class $\mathcal{C}$ and $n \geq 1$,

$$LP_{n+1}(\mathcal{C}) \vee RP_{n+1}(\mathcal{C}) = MPol(LP_n(\mathcal{C}) \vee RP_n(\mathcal{C})).$$

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$\Rightarrow$ Membership is decidable for **all** levels $LP_n(\mathcal{C}) \vee RP_n(\mathcal{C})$ as soon as this is the case for the first one (*i.e.*, $LP_1(\mathcal{C}) \vee RP_1(\mathcal{C})$).

Conclusion

The story ends here today (for a **group class** $\mathcal{G}$)

Logic	Operator
first-order logic: $\mathrm{FO}(<, \mathbb{P}_{\mathcal{G}})$	star-free closure: $\mathrm{SF}(\mathcal{G})$
$\mathcal{B}\Sigma_1(<, \mathbb{P}_{\mathcal{G}})$ and $\mathcal{B}\Sigma_1(<, +1, \mathbb{P}_{\mathcal{G}})$	$BPol(\mathcal{G})$ and $BPol(\mathcal{G}^+)$
two-variable first-order logic: $\mathrm{FO}^2(<, \mathbb{P}_{\mathcal{G}})$ and $\mathrm{FO}^2(<, +1, \mathbb{P}_{\mathcal{G}})$	$UPol(BPol(\mathcal{G}))$ and $UPol(BPol(\mathcal{G}^+))$
FO^2 hierarchy: all levels $\mathcal{B}\Sigma_n^2(<, \mathbb{P}_{\mathcal{G}})$ and $\mathcal{B}\Sigma_n^2(<, +1, \mathbb{P}_{\mathcal{G}})$	start from $BPol(\mathcal{G})$ and $BPol(\mathcal{G}^+)$, climb with $MPol$

In **all cases**, membership decidable if **$\mathcal{G}$-separation decidable**.

Thank you !