

**Next steps for implicit automata in λ -calculi:
vintage transducers, simple types and recursion schemes**

Lê Thành Dũng (Tito) NGUYỄN – joint work with Pierre PRADIC

ANR Delta closing meeting – June 1st, 2022

The big picture

Basic motivation: natural questions about the *expressiveness of typed λ -calculi*
(minimalistic functional programming languages)
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+ raise some speculative questions in pure automata theory

The λ -calculus and Church encodings

A naive syntactic theory of functions:

$$\begin{aligned} f x &\approx f(x) \\ \lambda x. t &\approx x \mapsto t \\ (\lambda x. t) u \rightarrow_{\beta} t\{x := u\} &\approx (x \mapsto x^2 + 1)(42) = 42^2 + 1 \end{aligned}$$

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The untyped λ -calculus is Turing-complete

The simply typed λ -calculus

We now consider a *type system*: labeling λ -terms with specifications

$$t : A \rightarrow B \quad \approx \quad "t \text{ is a function from } A \text{ to } B"$$

Simple types: built using " \rightarrow " from a base type o

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More generally, $t : \text{Nat} \iff \exists n \in \mathbb{N} : t =_{\beta\eta} \bar{n}$

Simply typed functions on Church numerals (1)

Simple types make the λ -calculus terminate: not Turing-complete anymore
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Theorem (Schwichtenberg 1975)

The functions $\mathbb{N}^k \rightarrow \mathbb{N}$ definable by simply-typed λ -terms $t : \text{Nat} \rightarrow \dots \rightarrow \text{Nat} \rightarrow \text{Nat}$ are the extended polynomials (generated by 0, 1, +, \times , id and ifzero).

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A trick to increase expressive power: for any simple type A , for $n \in \mathbb{N}$,

$$\bar{n} : \text{Nat}[A] = \text{Nat}\{o := A\} = (A \rightarrow A) \rightarrow A \rightarrow A$$

(but in general some inhabitants of $\text{Nat}[A]$ don't represent integers)

Open question

Choose some simple type A and some term $t : \text{Nat}[A] \rightarrow \text{Nat}$.

What functions $\mathbb{N} \rightarrow \mathbb{N}$ can be defined this way?

Simply typed functions on Church numerals (2)

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Why is nobody working on this seemingly natural question?

- Apparently, low hopes for a nice answer until now
 - you can express towers of exponentials
 - but not subtraction or equality (Statman 198X)
- Not so important for actual programming language theory
 - analogy: functional analysis for differential equations vs Banach space geometry for its own sake...

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Slogan: the above question is not PL theory, it's automata theory!

Defining languages in the simply typed λ -calculus

Church encodings of binary strings [Böhm & Berarducci 1985]

\simeq fold_right on a list of characters (generalizable to any alphabet; $\text{Nat} = \text{Str}_{\{1\}}$):

$$\overline{011} = \lambda f_0. \lambda f_1. \lambda x. f_0 (f_1 (f_1 x)) : \text{Str}_{\{0,1\}} = (o \rightarrow o) \rightarrow (o \rightarrow o) \rightarrow o \rightarrow o$$

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Theorem (Hillebrand & Kanellakis 1996)

All regular languages, and only those, can be defined this way.

Automata theory appears in the simply typed λ -calculus

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The language $L \subseteq \Sigma^*$ is regular \iff there are a simple type A and $t : \text{Str}_\Sigma[A] \rightarrow \text{Bool}$ such that $\forall w \in \Sigma^*, w \in L \iff t \bar{w} =_\beta \text{true}$

Corollary

A simply typed λ -term of type $\text{Str}_\Gamma[A] \rightarrow \text{Str}$ defined a function $f : \Gamma^* \rightarrow \Sigma^*$ which is regularity-preserving: $L \subseteq \Sigma^*$ regular $\implies f^{-1}(L)$ regular

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Another good property: these string-to-string functions are *closed under composition*
 \longrightarrow we might expect them to correspond to some *transducer* model!

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Another good property: these string-to-string functions are *closed under composition* \longrightarrow we might expect them to correspond to some *transducer* model!

However, these functions can have grow as fast as any tower of exponentials which is rarely the case for transducers (but precedents exist!)

So, we started out with a “strategic retreat”...

Implicit automata in linear logic

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Automata theory counterpart: various “single use restrictions”

Several machine models for *regular functions* of strings and trees involve such restrictions [Bloem & Engelfriet 2000; Engelfriet & Maneth 1999; Alur & Černý 2010; ...]

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\rightarrow λ -calculus characterizations of *regular* and *comparison-free polyregular* functions
+ star-free languages / aperiodic reg. fn. via non-commutative types
+ upcoming work on atoms (with Clovis Eberhart)

also relying on a single use restriction [Bojańczyk & Stefański 2020]

Streaming string transducers [Alur & Černý 2010]

DFA + string-valued *registers*. Example:

$$\begin{aligned} \text{mapReverse} : \{a, b, c, \#\}^* &\rightarrow \{a, b, c, \#\}^* \\ w_1\# \dots \#w_n &\mapsto \text{reverse}(w_1)\# \dots \#\text{reverse}(w_n) \end{aligned}$$

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a	c	a	b	$\#$	b	c	$\#$	c	a
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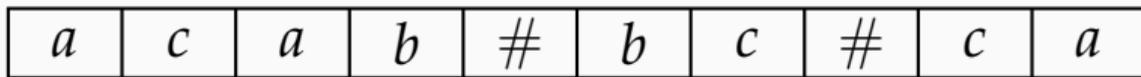
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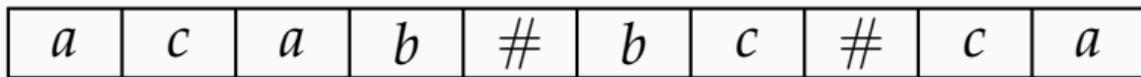
$$X = \textit{baca} \quad Y = \varepsilon$$

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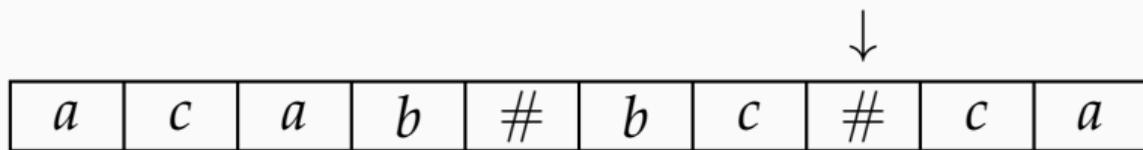
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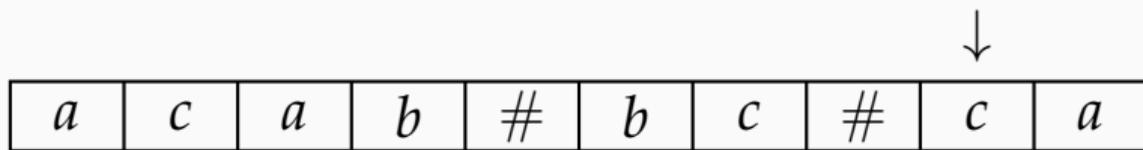


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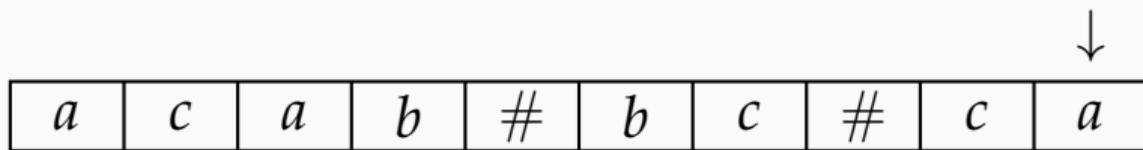


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Regular functions (a.k.a. MSO transductions) = computed by copyless SSTs

$$a \mapsto \begin{cases} X := aX \\ Y := Y \end{cases} \quad \# \mapsto \begin{cases} X := \varepsilon \\ Y := YX\# \end{cases} \quad \text{each register appears at most once on the right of a := in a transition}$$

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- can grow up to exponentially, e.g. $X := XX$

—→ not closed under composition; by composing we get towers of exp,
matching the known growth rate for simply typed λ -calculus

So, what is known about (compositions of) copyful SSTs?

What is known about (compositions of) copyful streaming string transducers?

Theorem (Filiot & Reynier 2017)

- *The much older HDT0L systems are isomorphic to “simple” copyful SSTs*
- *Copyful SSTs can be simplified → they compute HDT0L transductions*

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Next, let's search for this keyword in the literature...

Theorem (Ferté, Marin & Sénizergues 2014)

The following compute the same string-to-string functions:

- *another notion of HDT0L transduction = right-to-left (simple) copyful SSTs*
- *level-2 pushdown transducers: see next slide*

Pushdowns of pushdowns

Theorem (Ferté, Marin & Sénizergues 2014)

Right-to-left (simple) copyful SSTs \iff level-2 pushdown transducers

Let's compute $abc \mapsto (c)(bc)(abc)$

$\left[\begin{array}{c} \\ \\ \\ [abc] \end{array} \right]$

Output:

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Let's compute $abc \mapsto (c)(bc)(abc)$

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Output: cbc

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Output: $cbcabc$

Remark: we never need to push sth on the small stacks, they're input suffixes

\longrightarrow "one-way marble" transducers (à la [Douéneau-Tabot, Filiot & Gastin 2020])

Iterated pushdown transducers: using pushdowns of ... of pushdowns

We just saw the $k = 1$ case of:

Claim (Sénizergues 2007 — no available proof?)

Composition of k right-to-left copyful SSTs \iff level- $(k + 1)$ pushdown transducers

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Macro tree transducers [Engelfriet & Vogler 1985] can be seen as bottom-up automata with registers, generalizing right-to-left copyful SSTs to trees.

Theorem (Engelfriet & Vogler 1986 (note the different date))

*Composition of k macro tree transducers \iff level- k (not $k + 1$) pushdown transducers
manipulating pointers to the input tree
(provide input as pointer to root, not as stack of letters; pointers can only move downwards)*

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Note that this directly generalizes the “one-way marbles” ($k = 1$ on strings)

“Engelfriet’s class” of transductions

In fact, the following are equivalent: [Engelfriet & Vogler '88; Engelfriet & Maneth '03]

- Iterated pushdown tree transducers (with pointers)
- Compositions of macro tree transducers
 - of attribute grammars a.k.a. tree-walking transducers
 - of anything in-between (pebble transducers, MSOT w/ sharing, ...)
- “High level tree transducers”: can be viewed as storing *functions* in registers
 - (with subtle restrictions, we’ll come back to that)

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Trivial observation

They are included in the simply typed λ -definable functions.

But we’ll see why the converse might fail, via a detour through *infinite* structures

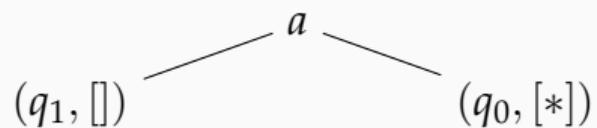
Generating infinite trees

Higher-order pushdown automata = iterated pushdown transducers without input

(q_0, \square)

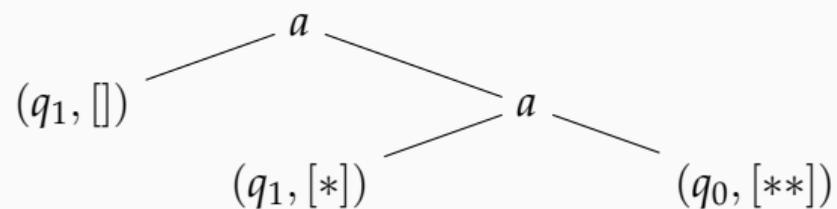
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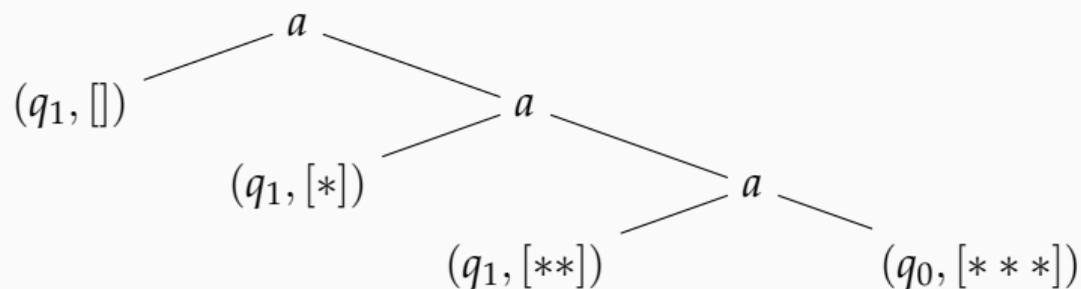
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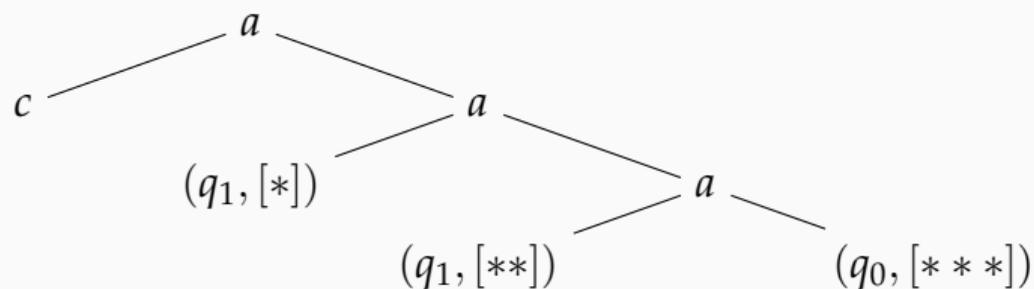
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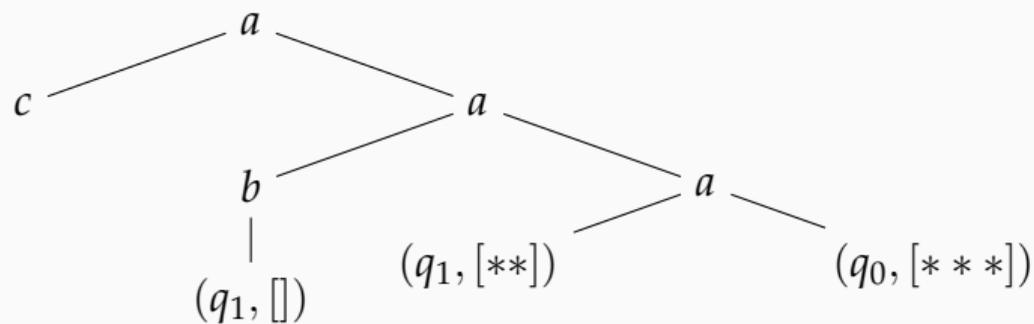
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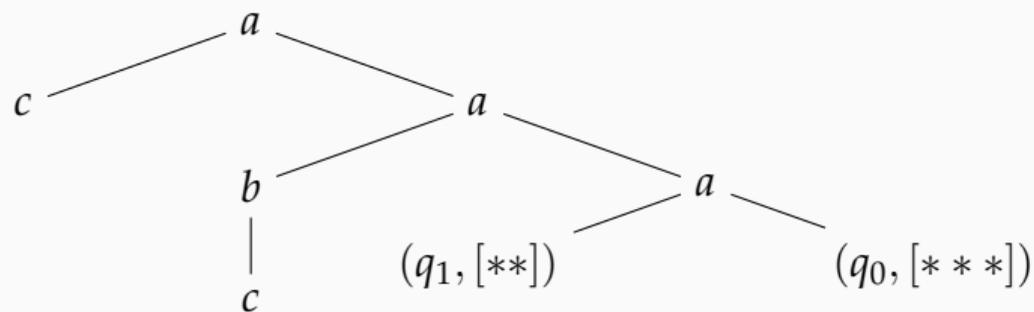
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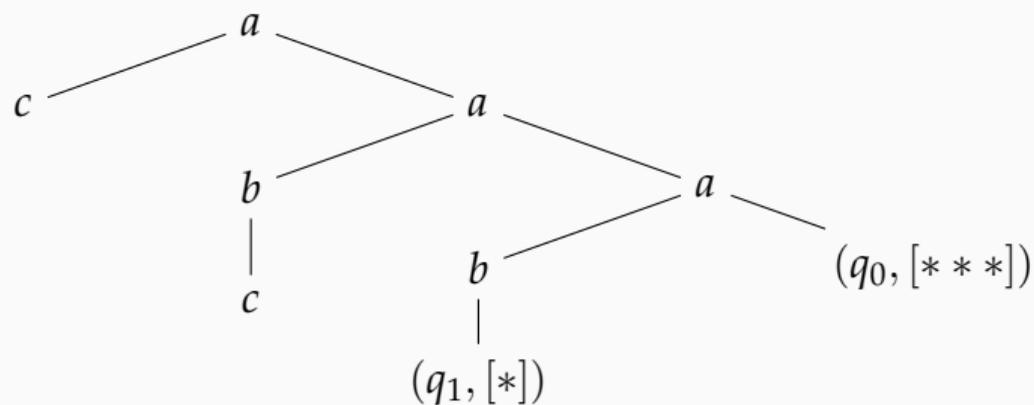
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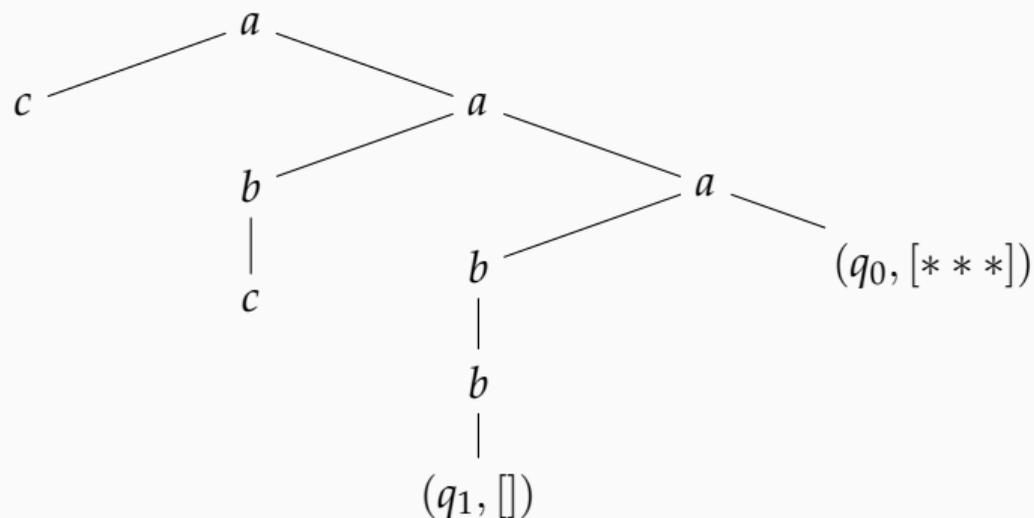
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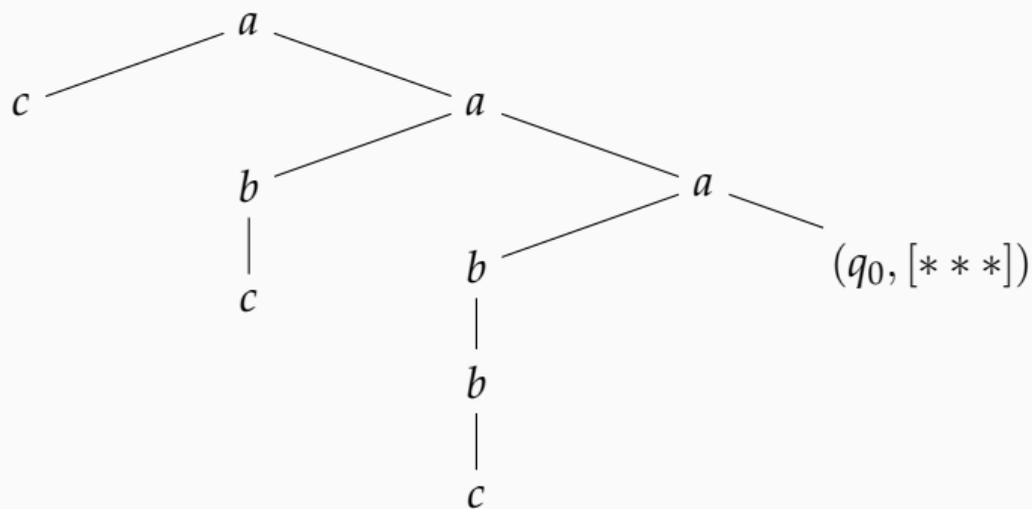
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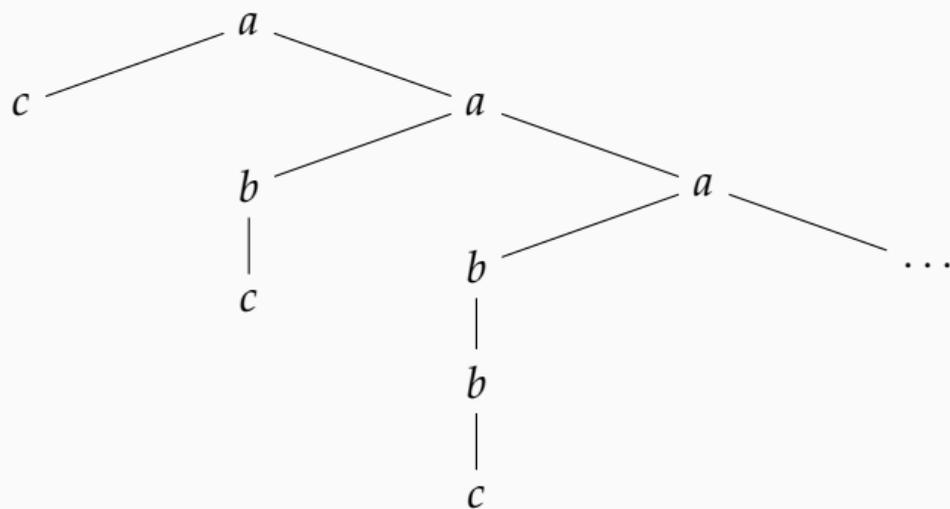
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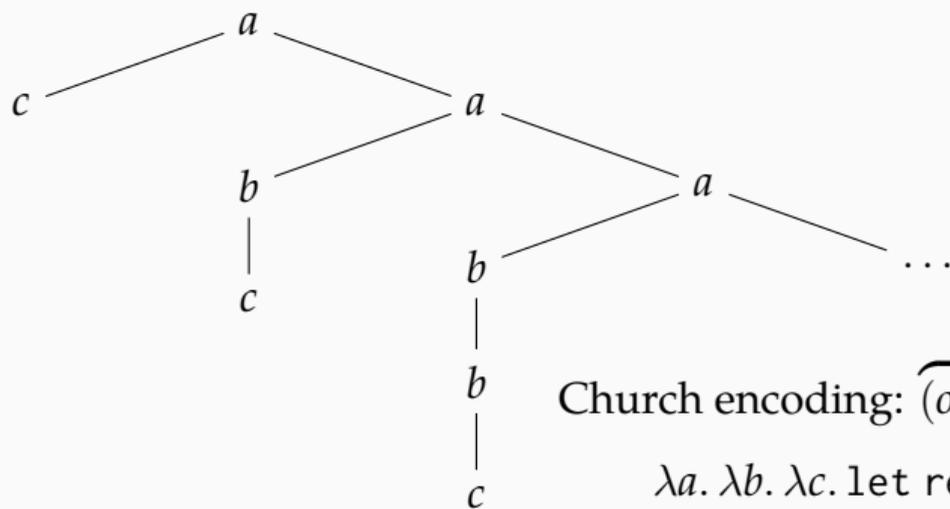
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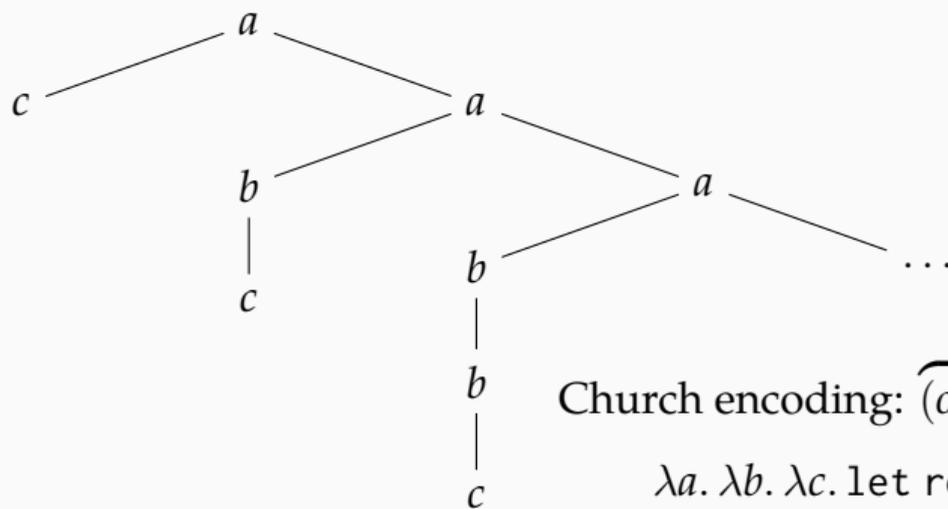


Church encoding: $\overbrace{(o \rightarrow o \rightarrow o)}^a \rightarrow \overbrace{(o \rightarrow o)}^b \rightarrow \overset{c}{\downarrow} o \rightarrow o$

$\lambda a. \lambda b. \lambda c. \text{let rec } f = \lambda x. a x (f(b x)) \text{ in } f c$

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Theorem (Damm '82; Knapkik, Niwiński & Urzyczyn '02; Salvati & Walukiewicz '12)

$\text{HOPDA} \iff$ so-called safe fragment of the simply typed λ -calculus with let rec

Equivalence for formalisms generating infinite trees

Higher-order pushdown automata \iff *safe* λ -calculus with let rec

- Safety was first introduced in another equivalent formalism, *recursion schemes*
- Engelfriet & Vogler's "high level tree transducers" are directly inspired from Damm's work on safe recursion schemes

Safely λ -definable functions

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\rightarrow **Claim**: the following should follow mostly routinely from previous work

Safe λ -terms (w/o let rec [Blum & Ong 2009]) of type $\text{Tree}_\Gamma[A] \rightarrow \text{Tree}_\Sigma$ compute the same functions as "high level TTs" / iterated pushdown transducers / ...

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But some trees can only be generated by *unsafe* recursion schemes [Parys 2012]
 \rightarrow safety could also decrease the λ -definable functions on finite trees

Collapsible pushdown transducers

Theorem (Hague, Murawski, Ong & Serre 2008)

Collapsible PDA generate the same trees as simply typed λ -terms with let rec

Additional structure on pushdowns of ... of pushdowns + collapse operation

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The “obvious” conjecture

The simply typed λ -definable functions (over Church encodings) are exactly those computable by some “collapsible pushdown tree transducer” model.

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- Engelfriet & Vogler’s proofs rely on inductive characterizations that are not available anymore in this setting...
- Technical issue: “collapsible pushdown transducers” can loop forever, the simply typed λ -calculus is terminating

Taking divergence into account

An attempt at decomposing the “obvious” conjecture

Let $f : \{\text{finite trees}\} \rightarrow \{\text{possibly infinite trees}\}$ be a partial function.

1. f is computed by a collapsible pushdown transducer
 $\iff f$ is defined by a simply typed λ -term with `let rec`
2. Furthermore, in that case, there is a simply typed λ -term *without* `let rec` defining a function that coincides with f on $f^{-1}(\{\text{finite trees}\})$.

Hopefully no unforeseen difficulty for (1), while (2) seems harder

Open question

Is there some “manifestly total” machine model for these functions?

More questions on simply typed λ -definable functions

- Can they be obtained by composing significantly simpler functions?
(recall that this works for the safe case i.e. iterated pushdown transducers)
- Does safety harm expressiveness over trees? over strings? over $\{a\}^* \cong \mathbb{N}$?
- Origin semantics using sets of ... of sets of input nodes?

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- Characterizations of subclasses by growth rate?

Theorem (Engelfriet, Inaba & Maneth 2021)

f computed by an iterated pushdown tree transducer $\wedge |f(t)| = O(|t|) \iff f$ is regular

Conjecture (Maximality of polyregular functions over strings)

f is simply typed λ -definable $\wedge |f(w)| = |w|^{O(1)} \iff f$ is polyregular

(i.e. a composition of polynomial growth HDTOL transductions, see [Bojańczyk 2018])

Conclusion

We started out by studying the functions definable in the simply typed λ -calculus
(on Church-encoded integers/strings/trees, with input type substitution)

- They (strictly?) include most (all?) known transduction classes, while still falling under the scope of automata theory (definable languages are regular)
- We conjectured a machine model & raised many questions
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Not the first time typed λ -calculi have led us to a new transducer model!

Most notably, discovery of comparison-free polyregular (or “polyblind”) functions, further studied by Douéneau-Tabot [N., Noûs & Pradic 2021]

Also: two-way transducers with planar behaviors for FO-transductions

a proposal for regular functions on trees with atoms using linear logic

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