

Determinization of transducers over real numbers

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Join work with A. Bès and Ch. Choffrut

Outline

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Expansion of real numbers in a fixed base b

Fix an integer base $b \geq 2$. The alphabet is $A = \{0, 1, \dots, b-1\}$.

- ▶ if $b = 2$, $A = \{0, 1\}$,
- ▶ if $b = 10$, $A = \{0, 1, 2, \dots, 9\}$.

Let $x \mapsto \hat{x}$ be the map from $A^{\mathbb{N}}$ to $[0, 1]$ defined by

$$\hat{x} = \widehat{x_1 x_2 \dots} = \sum_{k \geq 1} \frac{x_k}{b^k}.$$

$$\begin{aligned} A^{\mathbb{N}} &\rightarrow [0, 1] \\ x &\mapsto \hat{x} \\ 010101 \dots = (01)^{\mathbb{N}} &\mapsto 1/3 \\ 100000 \dots = 10^{\mathbb{N}} &\mapsto 1/2 \\ 011111 \dots = 01^{\mathbb{N}} &\mapsto 1/2 \\ 1100100100001111 \dots &\mapsto \pi/4 \end{aligned}$$

Consistency

The function f from $A^{\mathbb{N}}$ to $A^{\mathbb{N}}$ is **consistent** if

$$\hat{x} = \hat{x}' \quad \text{implies} \quad \widehat{f(x)} = \widehat{f(x')}.$$

It means that the following diagram commutes.

$$\begin{array}{ccc} A^{\mathbb{N}} & \xrightarrow{f} & A^{\mathbb{N}} \\ \downarrow \widehat{} & & \downarrow \widehat{} \\ [0, 1] & \xrightarrow{\widehat{f}} & [0, 1] \end{array}$$

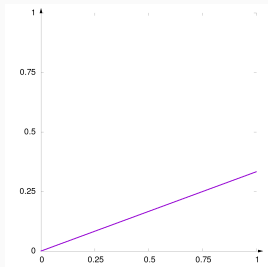
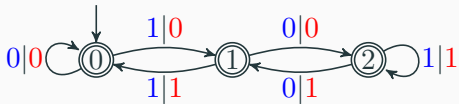
The function f defined by

$$f(x) = \begin{cases} 0^{\mathbb{N}} & \text{if } x \in 0A^{\mathbb{N}} \\ 1^{\mathbb{N}} & \text{if } x \in 1A^{\mathbb{N}} \end{cases}$$

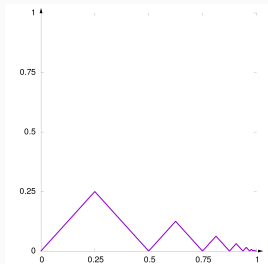
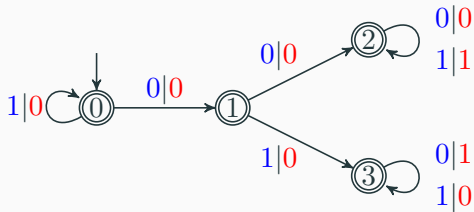
is **not** consistent: $\widehat{01^{\mathbb{N}}} = \widehat{10^{\mathbb{N}}} = \frac{1}{2}$ but $0 = \widehat{0^{\mathbb{N}}} \neq \widehat{1^{\mathbb{N}}} = 1$.

Example 1: division by 3

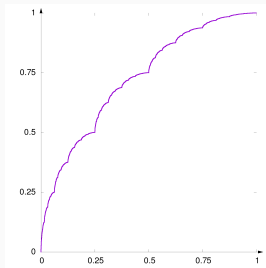
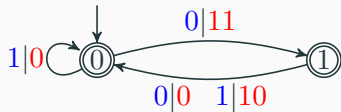
$$\hat{f}(x) = \frac{x}{3}$$



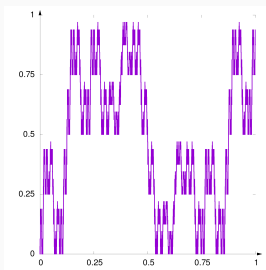
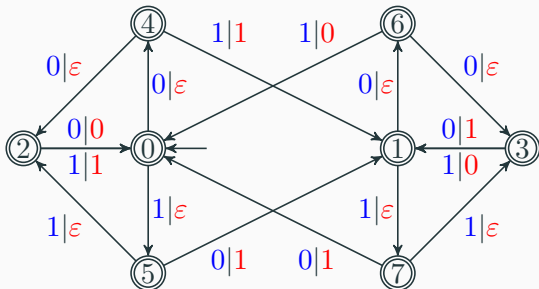
Example 2: fractal-like function



Example 3: function with zero derivative



Example 4: function with non-bounded variation



Outline

Real number functions realized by transducers

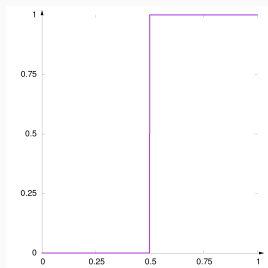
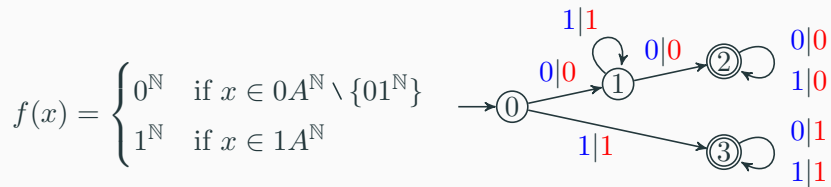
Continuity

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Continuity: step function

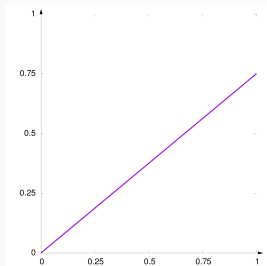
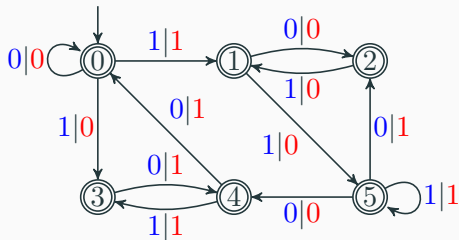
The continuity of f **does not** imply the continuity of \hat{f} .



Continuity: multiplication by 3

The continuity of \hat{f} **does not** imply the continuity of f .

$$\hat{f}(x) = \frac{3x}{4}$$



Continuity: multiplication by 3

- ▶ If $x <_{\text{lex}} (01)^{\mathbb{N}}$, that is, $\hat{x} < 1/3$, then $\hat{f}(x) < 1/4$ and thus $f(x) \in 00A^{\mathbb{N}}$.
- ▶ If $x >_{\text{lex}} (01)^{\mathbb{N}}$, that is, $\hat{x} > 1/3$, then $\hat{f}(x) > 1/4$ and thus $f(x) \notin 00A^{\mathbb{N}}$.
- ▶ f is **not continuous** at $x_0 = (01)^{\mathbb{N}}$.
- ▶ f cannot be realized by an input deterministic transducer.

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Avižienis numeration system

Fix a base $b \geq 3$ and a set of digits $A = \{-a, \dots, -1, 0, 1, \dots, a\}$ where $b/2 < a < b$.

For instance $b = 3$ and $A = \{-2, -1, 0, 1, 2\} = \{\bar{2}, \bar{1}, 0, 1, 2\}$

$$\widehat{d_n \cdots d_0} = \sum_{k=0}^n d_k b^k \quad \text{where } d_k \in A$$

Base 3 expansion	Decimal value
102	11
11 $\bar{1}$	11
2 $\bar{2}\bar{1}$	11
1 $\bar{1}\bar{2}\bar{1}$	11
1 $\bar{2}\bar{0}\bar{2}$	11
1 $\bar{2}\bar{1}\bar{1}$	11

Properties of Avīžienis numeration system

- ▶ 0 has exactly one expansion because $a < b$,
- ▶ non-zero integers have several expansions because $b/2 < a$,
- ▶ non-zero integers have infinitely many expansions if $a = b - 1$,
- ▶ taking the opposite is easy,
- ▶ addition can be performed **in parallel in constant time**.

Addition in Avižienis numeration system

Suppose

$$x = \sum_{k=0}^n x_k b^k \qquad y = \sum_{k=0}^n y_k b^k$$
$$z = x + y = \sum_{k=0}^n z_k b^k$$

where

► For each k in parallel do

1. $z_k := x_k + y_k$
2. if $-a < z_k < a$ then $c_{k+1} := 0$
3. if $z_k \geq a$ then $z_k := z_k - b$, $c_{k+1} := 1$
4. if $z_k \leq -a$ then $z_k := z_k + b$, $c_{k+1} := -1$
5. $z_k := z_k + c_k$

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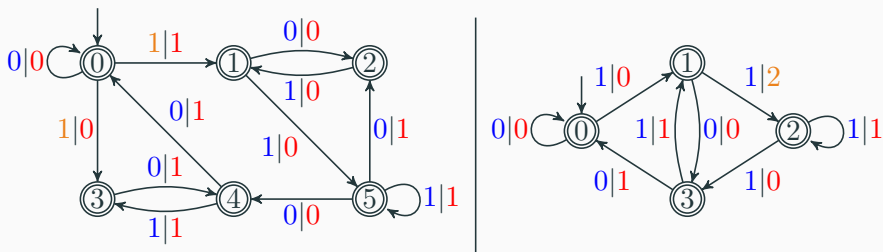
Avižienis numeration system

Back to transducers

Multiplication by 3 strikes back with more digits

Two transducers realizing the function $\hat{f} : x \mapsto \frac{3x}{4}$ in **base 2**.

- ▶ The one on the left is **not input deterministic** and the output alphabet is $\{0, 1\}$.
- ▶ The one on the right is **input deterministic** and the output alphabet is $\{0, 1, 2\}$.



Sufficient condition

Theorem

Let b be a base and \mathcal{T} be a transducer with input and output alphabet $A = \{0, \dots, b-1\}$ realizing a continuous function \hat{f} such that

$$\begin{array}{l} \text{if} \\ \rightarrow i \xrightarrow{u_1|v_1} q \curvearrowright u_2|v_2 \\ \rightarrow i' \xrightarrow{u_1|v'_1} q' \curvearrowright u_2|v'_2 \end{array} \quad \text{in } \mathcal{T} \text{ then } |v_2| = |v'_2|.$$

Then there exists some integer $b' \geq b$ and an input deterministic transducer \mathcal{T}' with input alphabet $\{0, \dots, b-1\}$ and output alphabet $\{0, \dots, b'-1\}$ which realizes \hat{f} .

Full characterization

A state is **constant** if the function realized from it is constant.

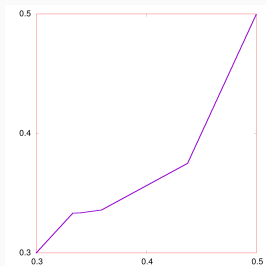
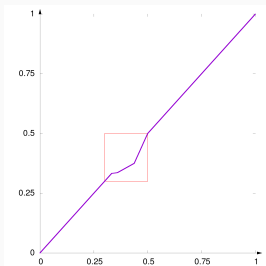
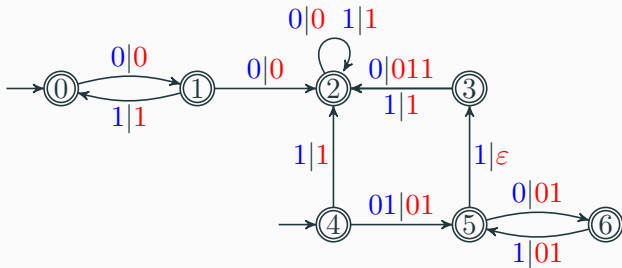
Theorem

Let b be a base and \mathcal{T} be a transducer with input and output alphabet $A = \{0, \dots, b-1\}$ realizing a continuous function \hat{f} such that for all **non-constant states** q and q'

$$\text{if } \begin{array}{l} \rightarrow i \xrightarrow{u_1|v_1} q \curvearrowright u_2|v_2 \\ \rightarrow i' \xrightarrow{u_1|v'_1} q' \curvearrowright u_2|v'_2 \end{array} \text{ in } \mathcal{T} \text{ then } |v_2| = |v'_2|.$$

Then there exists some integer $b' \geq b$ and an input deterministic transducer \mathcal{T}' with input alphabet $\{0, \dots, b-1\}$ and output alphabet $\{0, \dots, b'-1\}$ which realizes \hat{f} . **The converse holds.**

Example: transducer not satisfying the condition



Comparison with functions from words to words

Theorem (Choffrut 77)

A function from A^* to B^* realized by a transducer \mathcal{T} can be realized by an input deterministic transducer iff

$$\begin{array}{l} \text{if} \\ \rightarrow i \xrightarrow{u_1|v_1} q \curvearrowright u_2|v_2 \\ \rightarrow i' \xrightarrow{u_1|v'_1} q' \curvearrowright u_2|v'_2 \end{array} \quad \text{in } \mathcal{T} \text{ then} \quad \begin{array}{l} |v_2| = |v'_2| \\ \text{and} \\ v_1 v_2^\omega = v'_1 v'_2{}^\omega. \end{array}$$

Theorem (Béal-C. 02)

A function from $A^{\mathbb{N}}$ to $B^{\mathbb{N}}$ realized by a transducer \mathcal{T} can be realized by an input deterministic transducer iff for all *non-constant states* q, q'

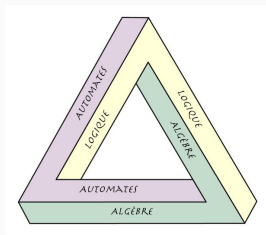
$$\begin{array}{l} \text{if} \\ \rightarrow i \xrightarrow{u_1|v_1} q \curvearrowright u_2|v_2 \\ \rightarrow i' \xrightarrow{u_1|v'_1} q' \curvearrowright u_2|v'_2 \end{array} \quad \text{in } \mathcal{T} \text{ then} \quad \begin{array}{l} |v_2| = |v'_2| \\ \text{and} \\ v_1 v_2^\omega = v'_1 v'_2{}^\omega. \end{array}$$

Open questions

- ▶ Characterization based on \hat{f} and not on the transducer realizing f .
- ▶ Extension to two-way transducers.

Farewell

- ▶ Merci d'avoir participé à cette réunion DeLTA
- ▶ Bon retour à tous.



Merci