

Decidable Logics for
Transductions and Data Words

Joint work with:

Luc Dartois and Emmanuel Filiot

Motivations

- ▶ High level formalism
- ▶ Model-checking: $T \models \varphi$

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Existing logical formalism: MSOT (Courcelle)

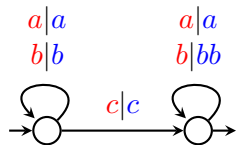
Approach

Transductions: subsets of $\Sigma^* \times \Sigma^*$

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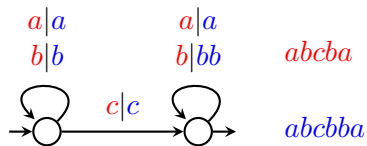
Example



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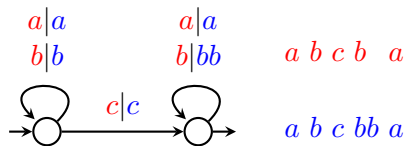
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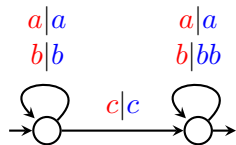
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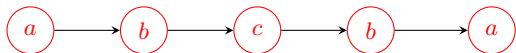
$a b c b a$

$a b c bb a$

origin information

Logical transductions

MSO on labelled graphs



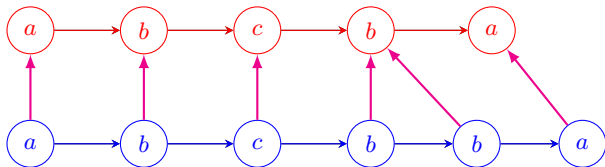
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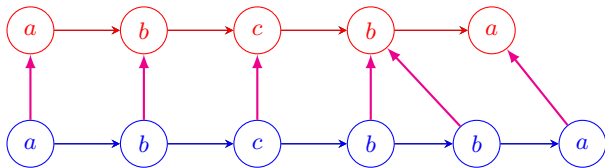
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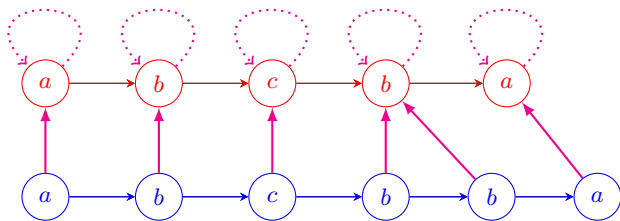
MSO on labelled graphs



MSO[$<, <, o$]

Logical transductions

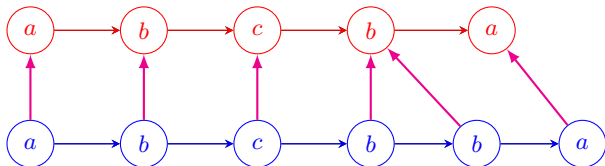
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$\text{MSO}[\prec, \ll, \circ]$

Logical transductions

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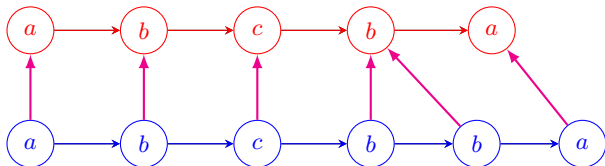
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Example

$$\begin{aligned} & \forall x \bigwedge_{\sigma \in \Sigma} (\sigma(x) \leftrightarrow \sigma(\circ(x))) \\ \blacktriangleright \phi_{\text{shuffle}} = & \bigwedge \forall x, y \circ(x) = \circ(y) \rightarrow x = y \\ & \bigwedge \forall x \exists y \circ(y) = x \end{aligned}$$

Logical transductions

MSO on labelled graphs



MSO[$<$, $<$, \circ]

Example

- $\forall x \bigwedge_{\sigma \in \Sigma} (\sigma(x) \leftrightarrow \sigma(\circ(x)))$
- ▶ $\phi_{\text{shuffle}} = \bigwedge \forall x, y \circ(x) = \circ(y) \rightarrow x = y$
 $\bigwedge \forall x \exists y \circ(y) = x$
- ▶ $\phi_{\text{mirror}} = \phi_{\text{shuffle}} \bigwedge \forall x, y (x < y) \leftrightarrow (\circ(y) < \circ(x))$

Logical transductions

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Undecidable satisfiability

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$$\varphi ::= \exists x \varphi \mid \varphi \vee \varphi \mid \neg \varphi \mid s \prec t \mid \sigma(s) \mid P(s, t) \quad (s, t \text{ terms})$$

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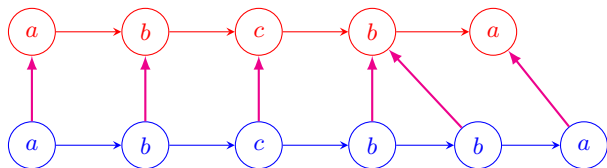
$$P ::= \exists x P \mid \exists X P \mid P \vee P \mid \neg P \mid X(x) \mid x \prec y \mid \sigma(x)$$

Results

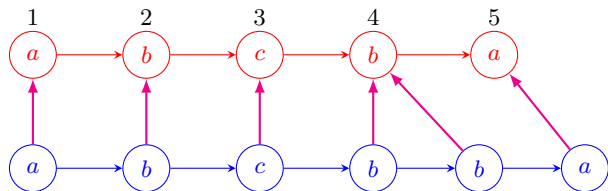
- ▶ Any MSOT is expressible in \mathcal{L}_T
- ▶ $\mathcal{L}_T = \text{FO}^2[\prec, \circ, \text{MSO}[\Sigma, \prec]]$ has decidable satisfiability

Data words and transductions

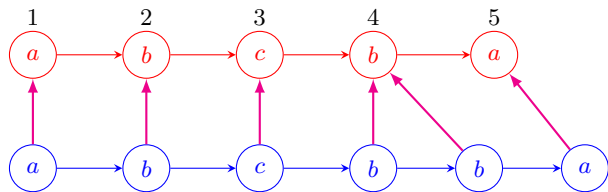
Data words and transductions



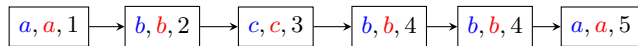
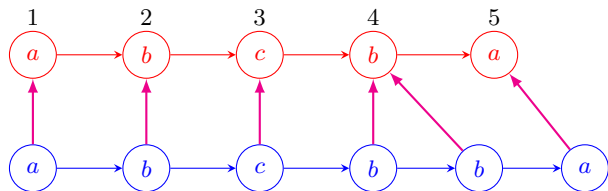
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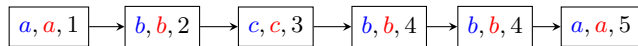
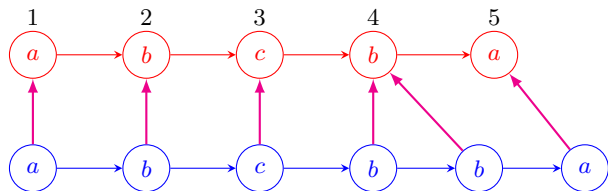
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Typed Data words:

$(\Sigma \times \Sigma \times \mathbb{N})^*$

with $w = (a_1, t_{d_1}, d_1) \dots (a_n, t_{d_n}, d_n)$

Data word logics

Previous logics

$\text{FO}^2[\prec, S_{\prec}, \prec]$ U

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$\text{FO}^2[\langle, \prec, \mathcal{S}_{\prec}] \text{ D}$

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→ extend this result to

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Our logic:

$$\mathcal{L}_D = \text{FO}^2[\prec, \text{MSO}[\Sigma, \prec]]$$

(Quantifications in MSO restricted to sets "compatible" with \sim)

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Equivalence

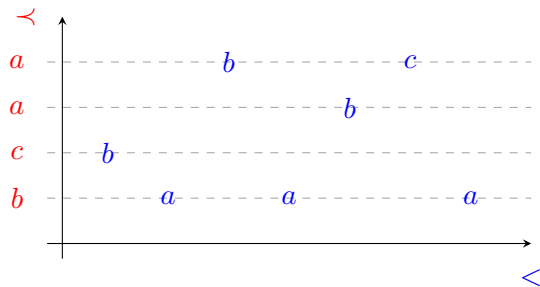
Decidability of \mathcal{L}_T reduces to decidability of \mathcal{L}_D and *vice versa*.

Data words as 2D sets of points

$(b, c, 2)(a, b, 1)(b, a, 4)(a, b, 1)(b, a, 3)(c, a, 4)(a, b, 1)$

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SNF and constraints

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Scott Normal Form \rightarrow conjunction of $\forall x \forall y$ and $\forall x \exists y$ formulas

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Constraints

- ▶ Universal constraint:

$$\sigma(x) \wedge \gamma(y) \wedge x \leq y \wedge P(x, y)$$

- ▶ Existential constraint:

$$\sigma(x) \rightarrow \exists y \bigvee_{\gamma, \leq, P} \gamma(y) \wedge x \leq y \wedge P(x, y)$$

Profile of a line

Abstraction of a horizontal line

Profile

A type σ and a sequence of clauses:

- ▶ (\cdot, σ)
- ▶ $(\uparrow P, \sigma)$
- ▶ $(\downarrow P, \sigma)$
- ▶ No more than twice a clause

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Results

- ▶ Model \leftrightarrow consistent sequence of profiles
- ▶ Construct an automaton recognizing consistent profile sequences

What we can decide

- ▶ Satisfiability
- ▶ Bounded origin
- ▶ Bounded-origin uniformization
- ▶ Functionality

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And for data words too !



30

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CAR CARE

CAR CARE