Universal algorithms for parity games and nested fixpoints

Anr Delta meeting

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June 28, 2021, in Paris!

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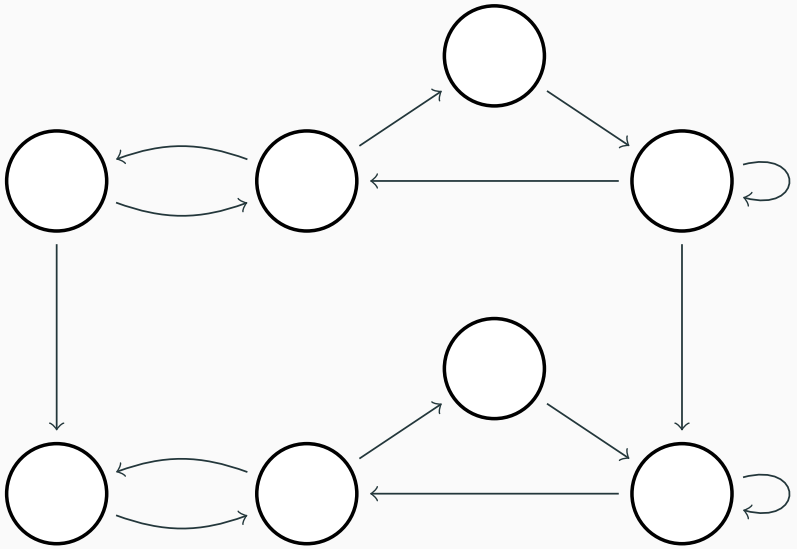
[2] École normale supérieure Paris-Saclay

Parity games



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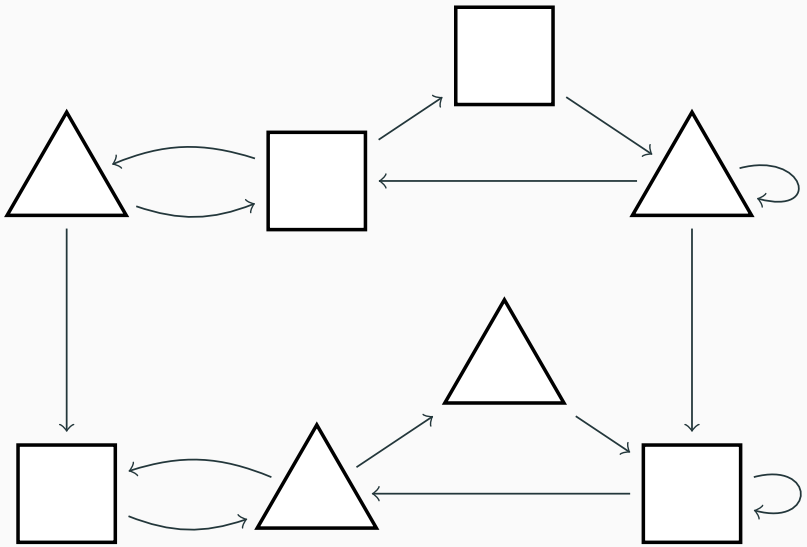

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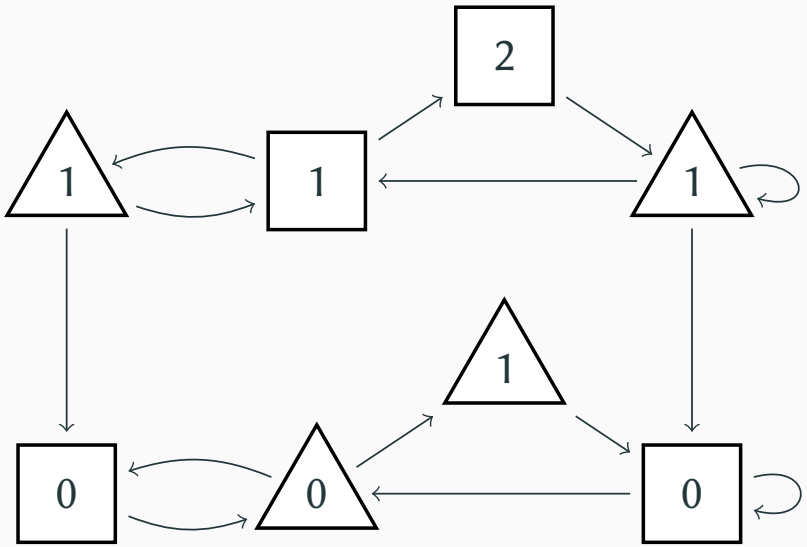

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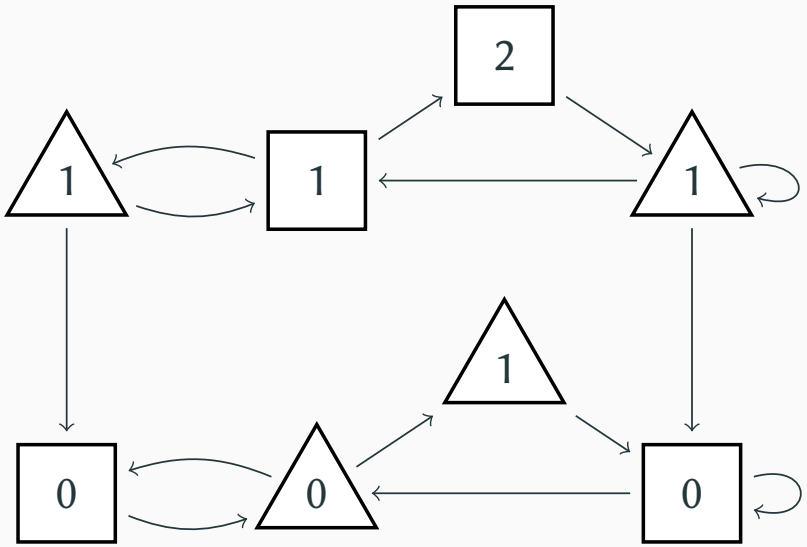

- Play: $(v_i)_{i \in \mathbb{N}}$ s.t. $\forall i,\ (v_i, v_{i+1}) \in E$.

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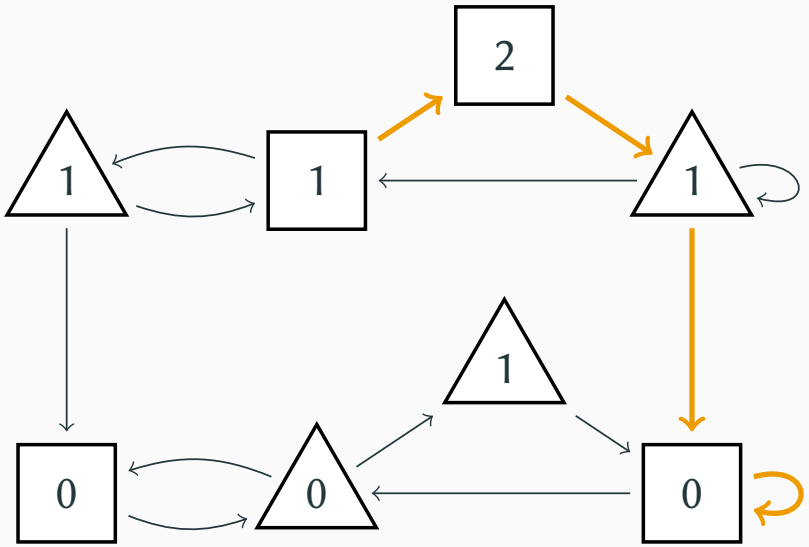

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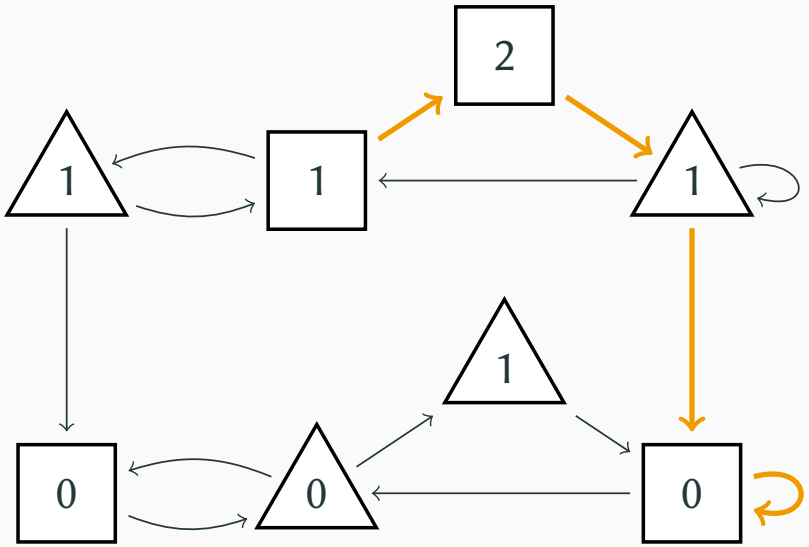

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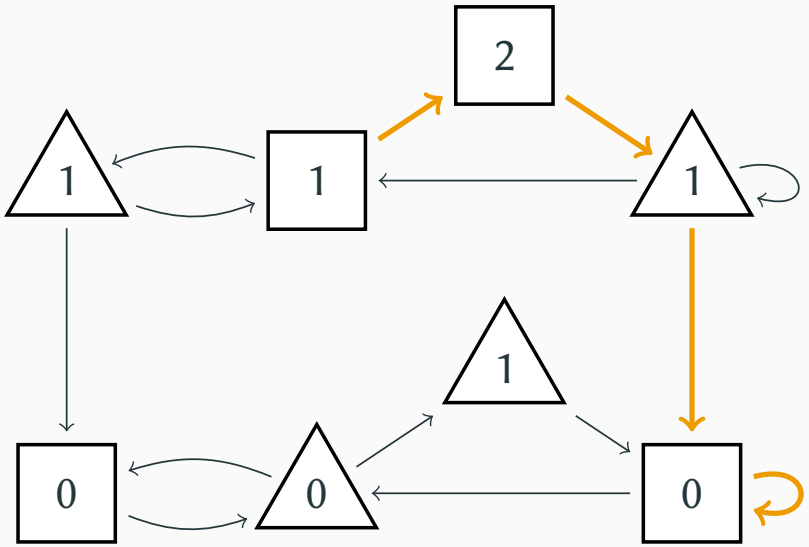

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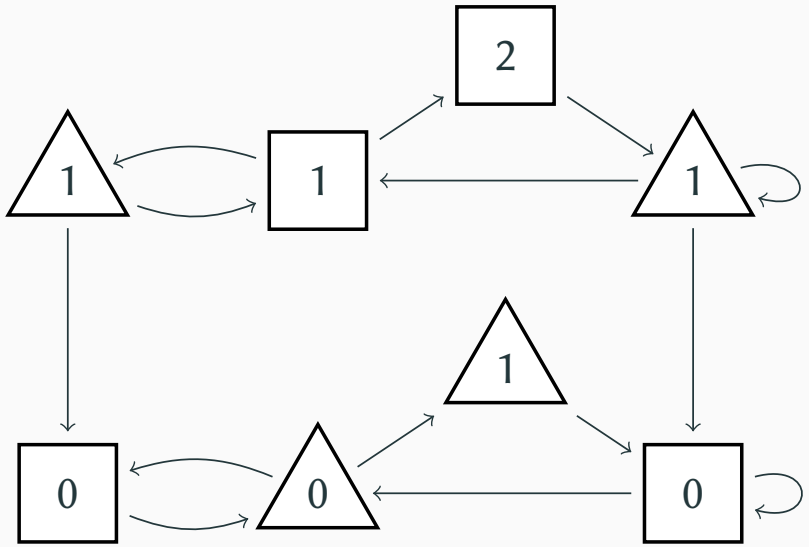

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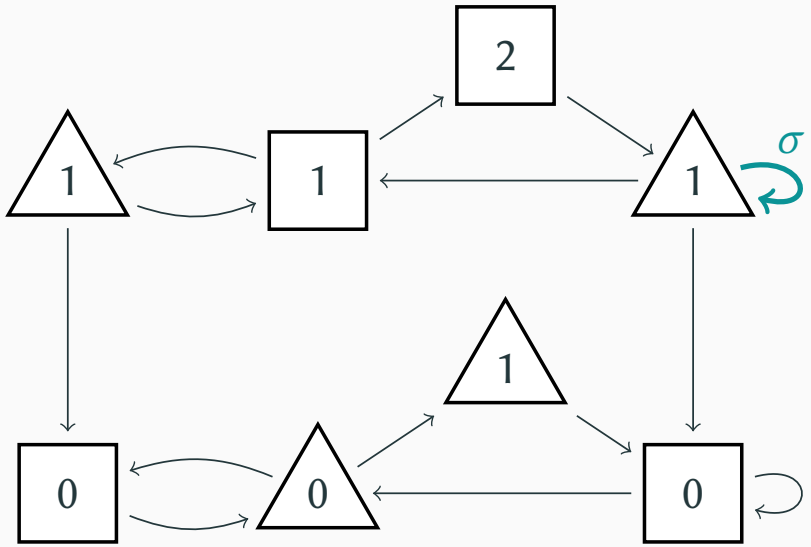

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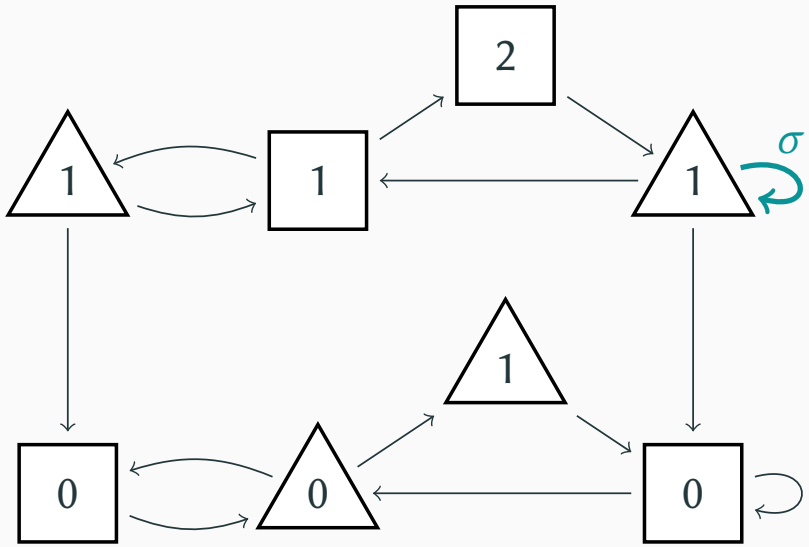

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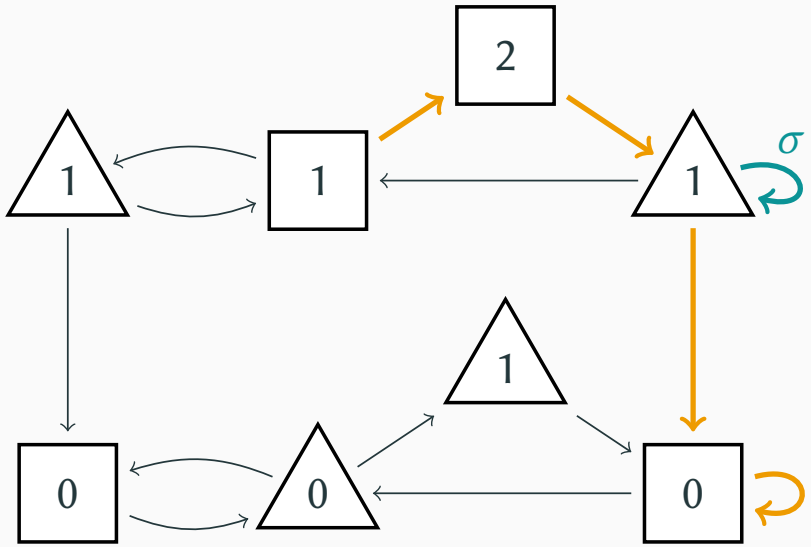

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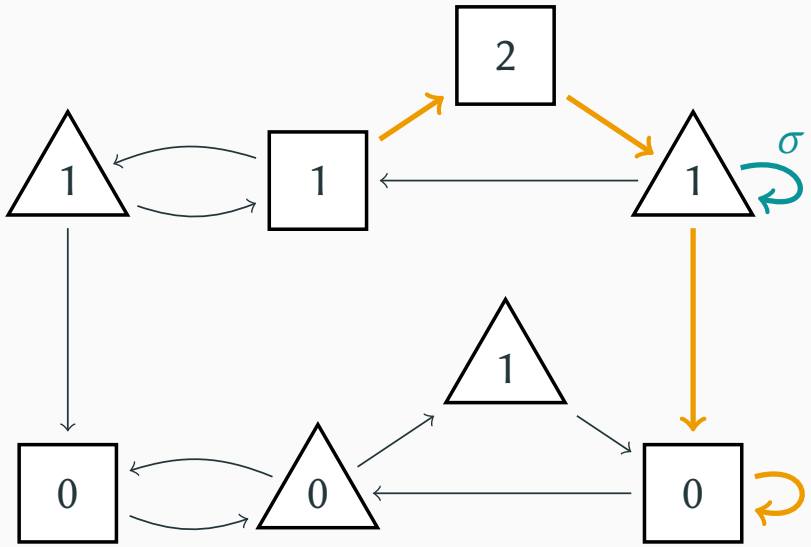

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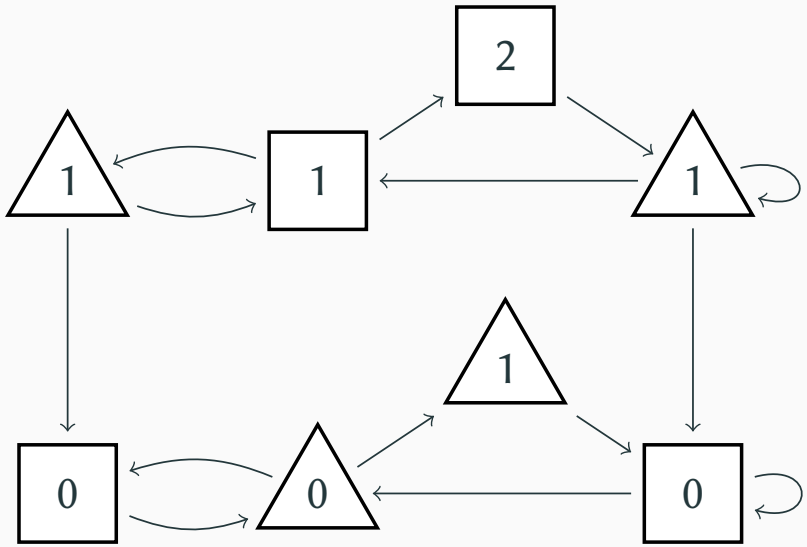

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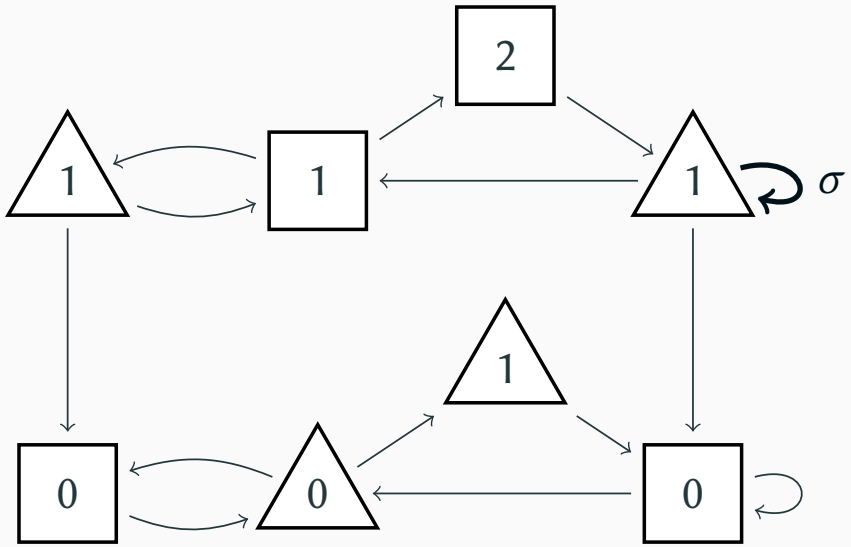

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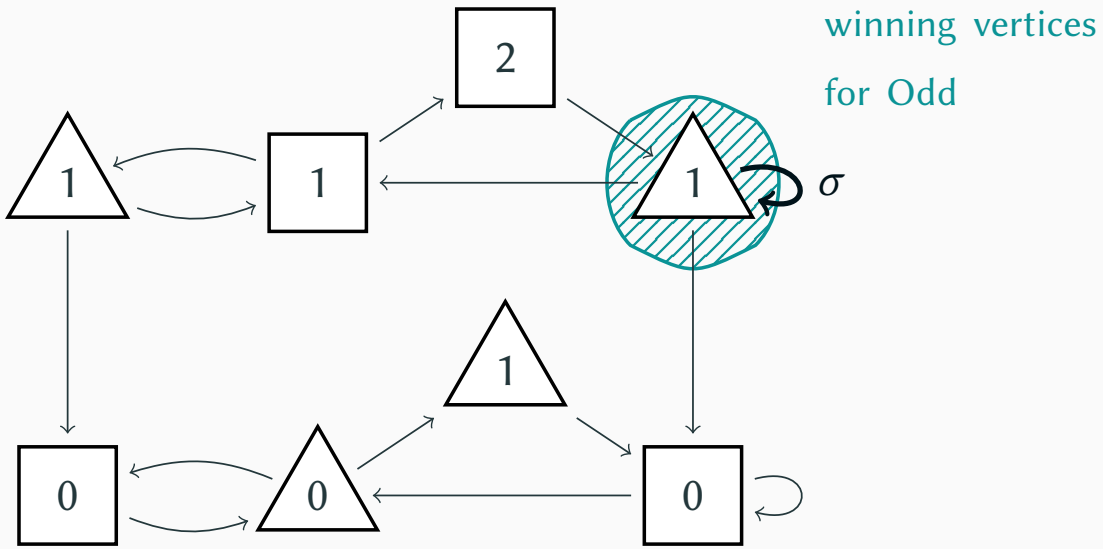

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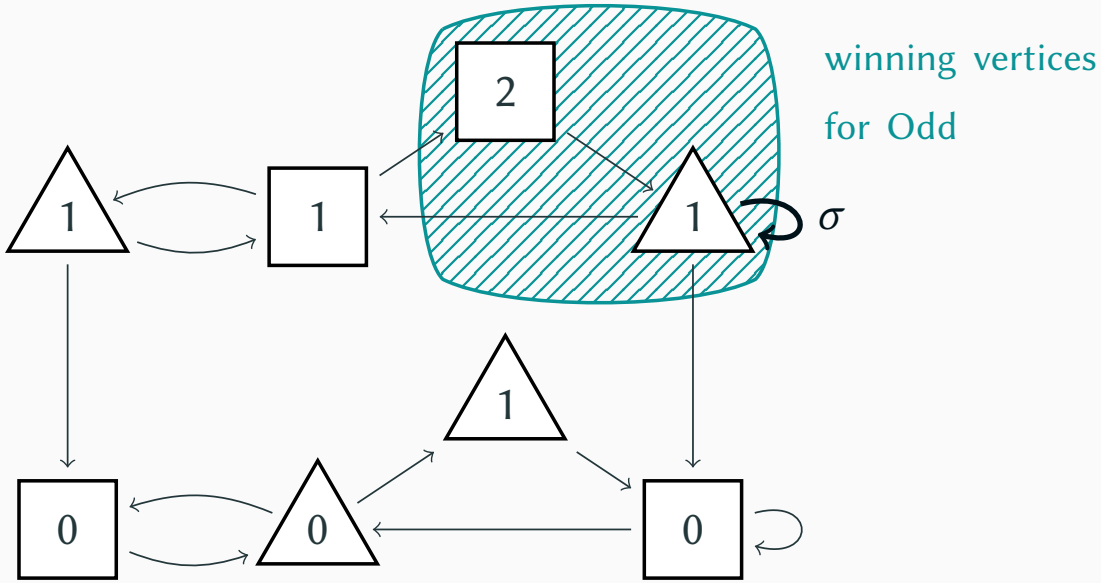

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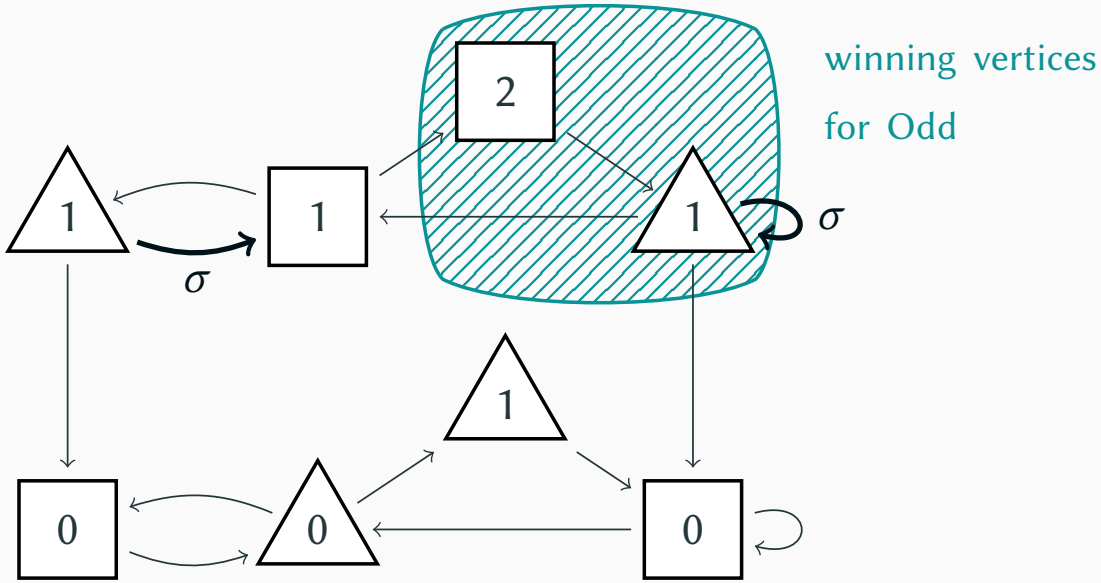

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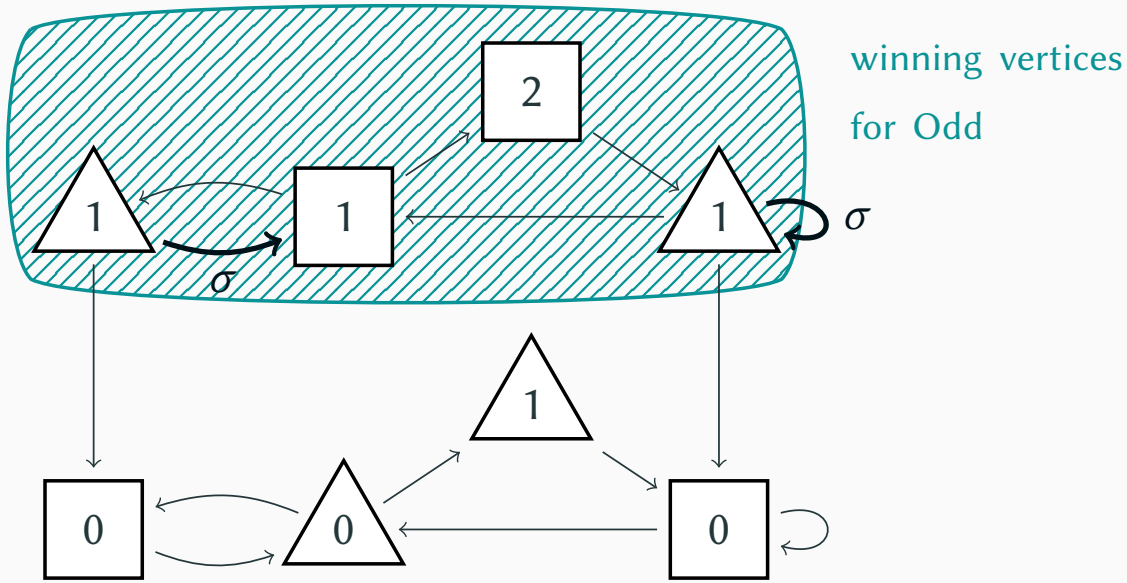

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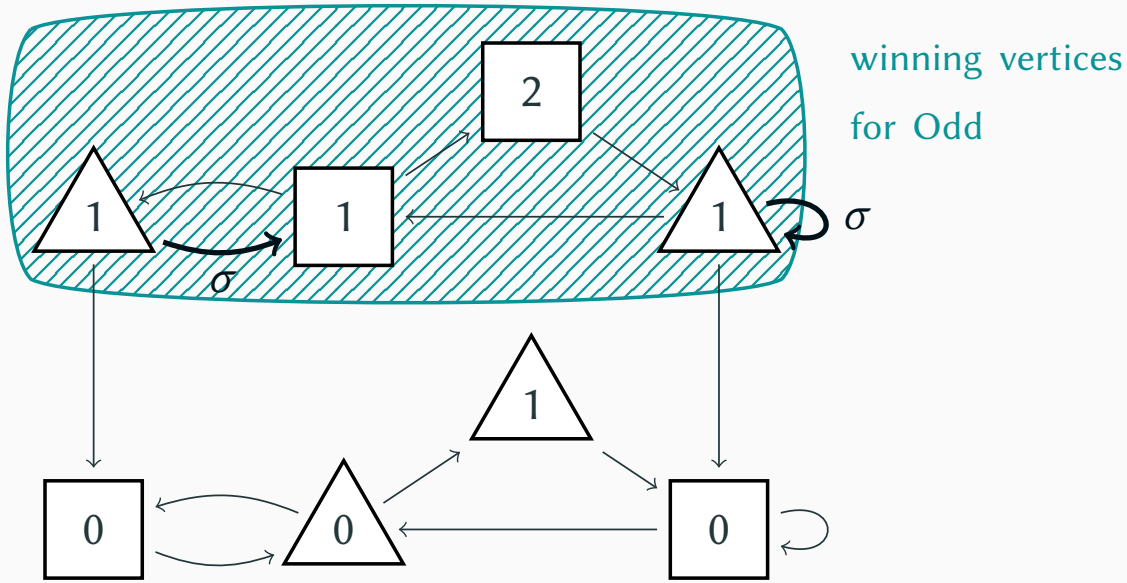

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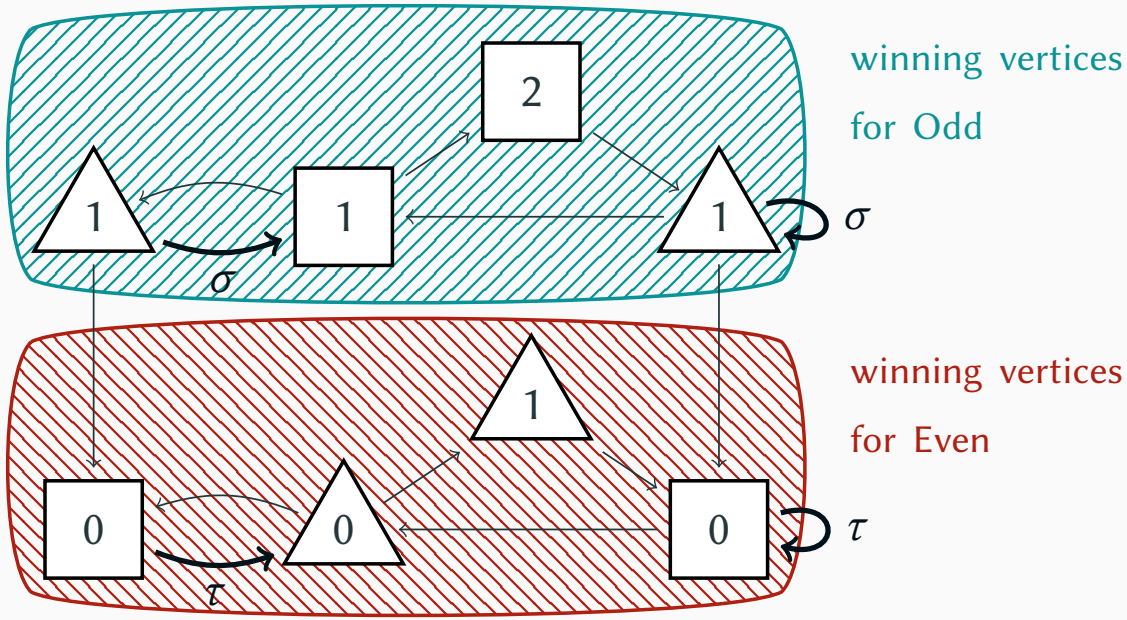

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Inputs: $\mathcal{G}$: parity game,

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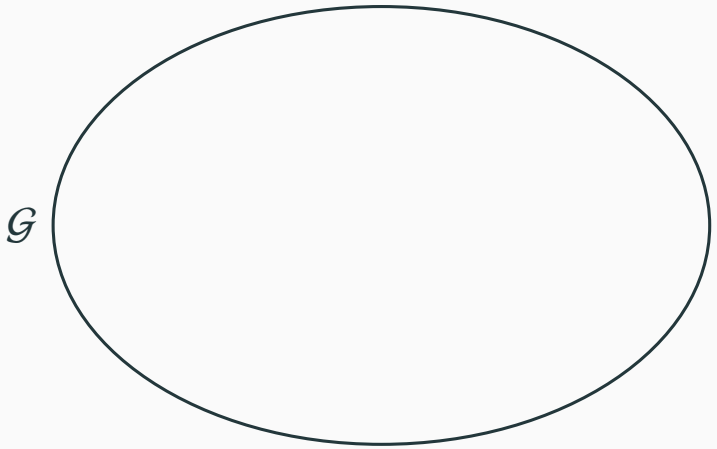
- Best known upper bound: quasipolynomial time $O(n^{\log(d)})$ ['17 Calude-Jain-Khoussainov-Li-Stephan]

A recursive algorithm?



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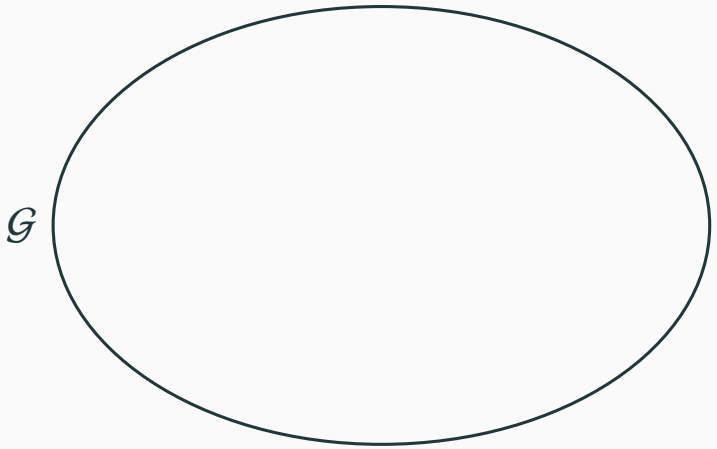

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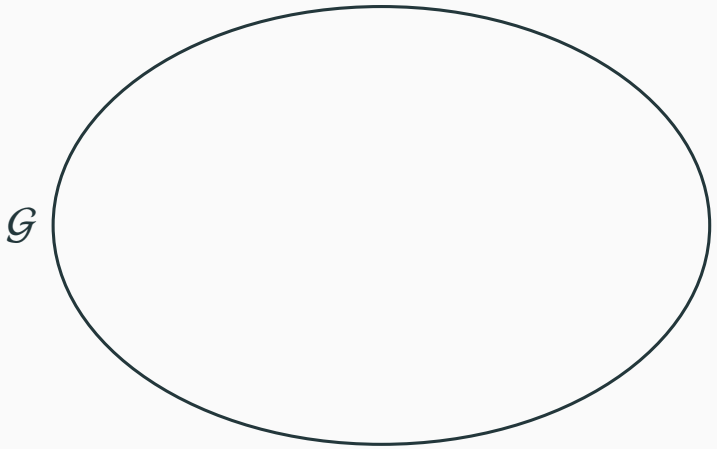

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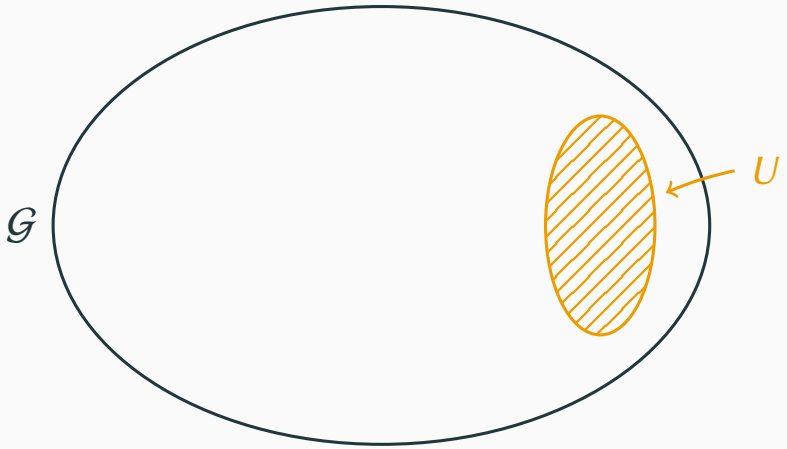

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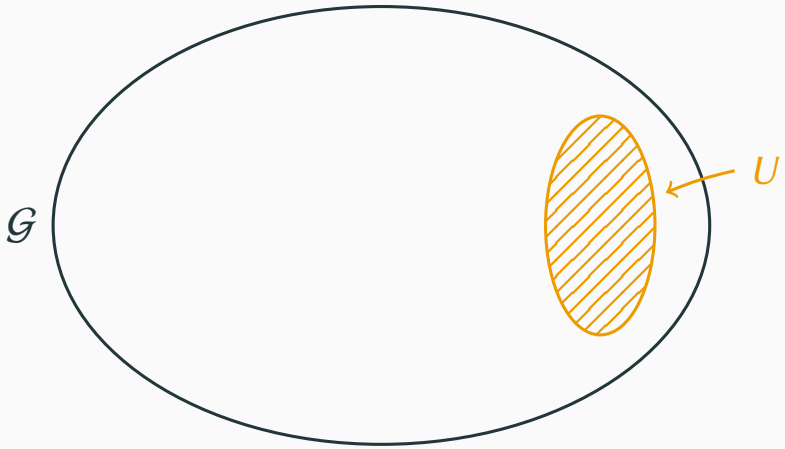

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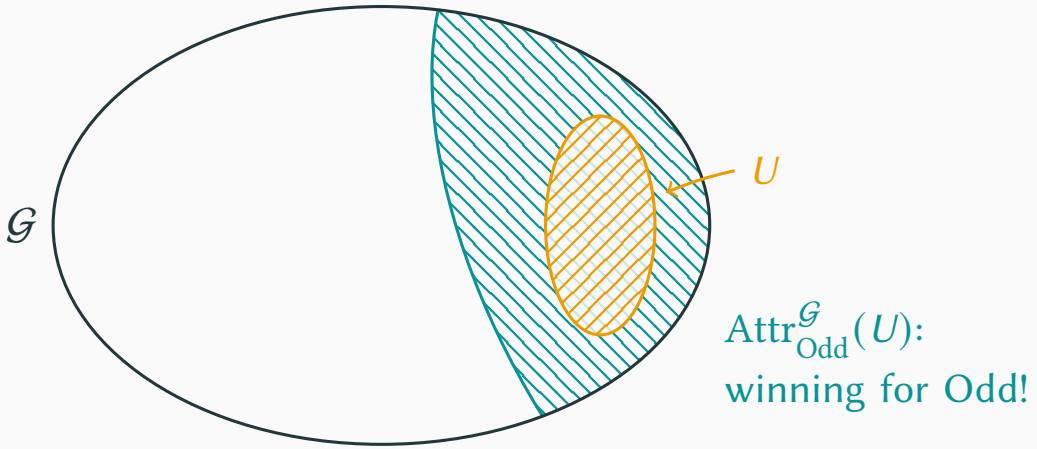

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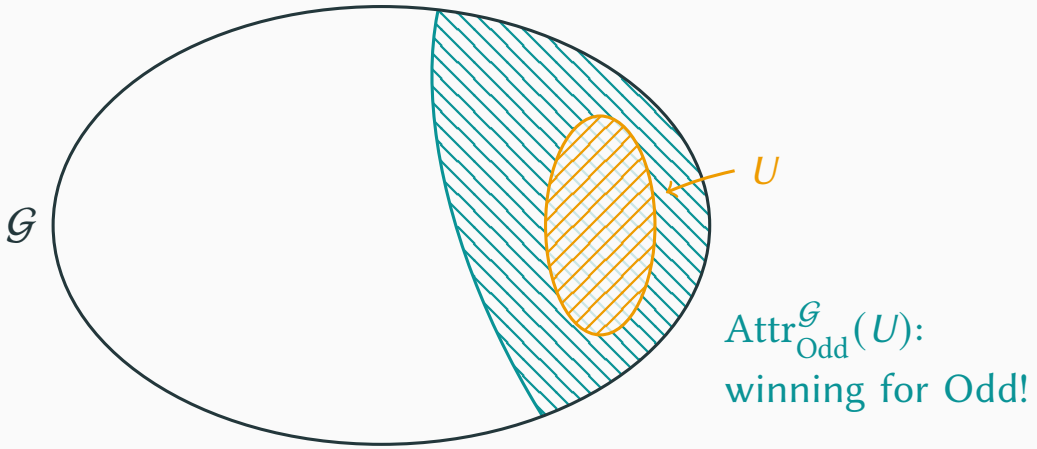

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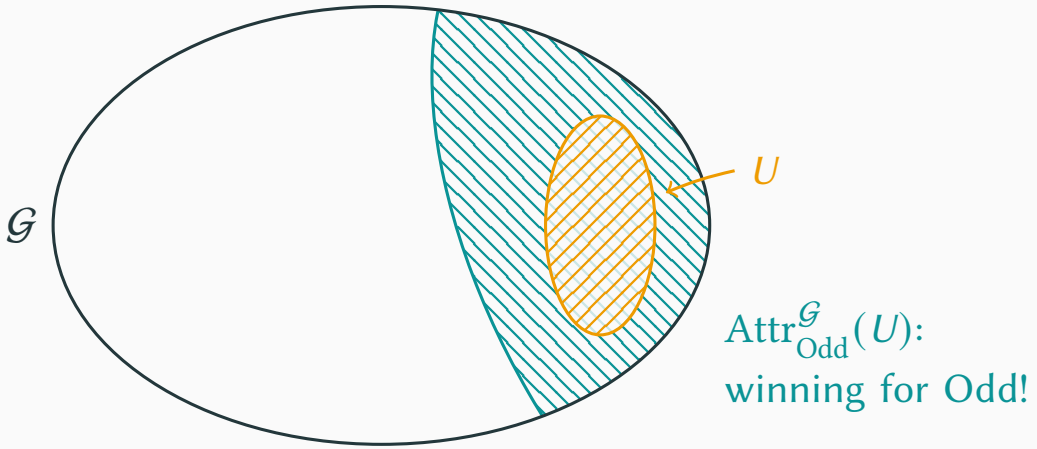

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How to compute the set of v s.t. Odd can win from v without ever seeing a vertex of priority d? It is the set of winning vertices for Odd in the game

$$\mathcal{G} \setminus \mathrm{Attr}^{\mathcal{G}}_{\mathrm{Even}}(\pi^{-1}[d]).$$

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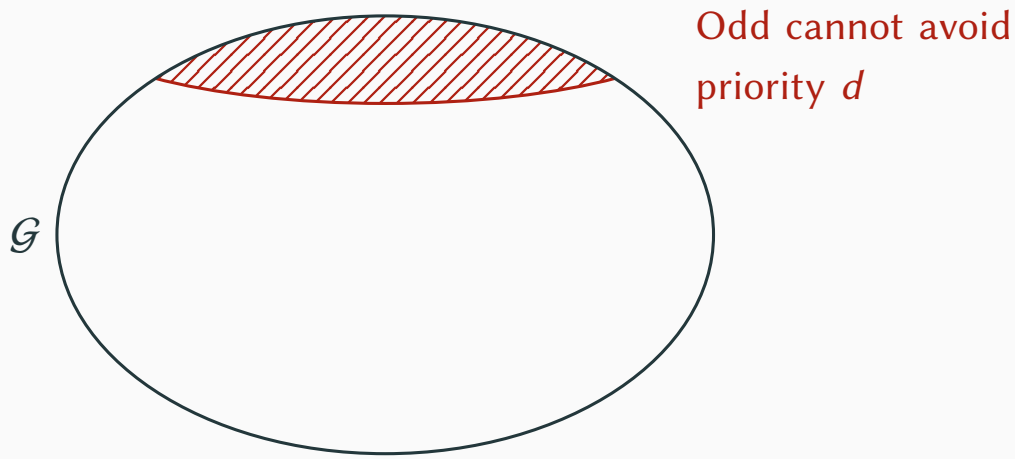

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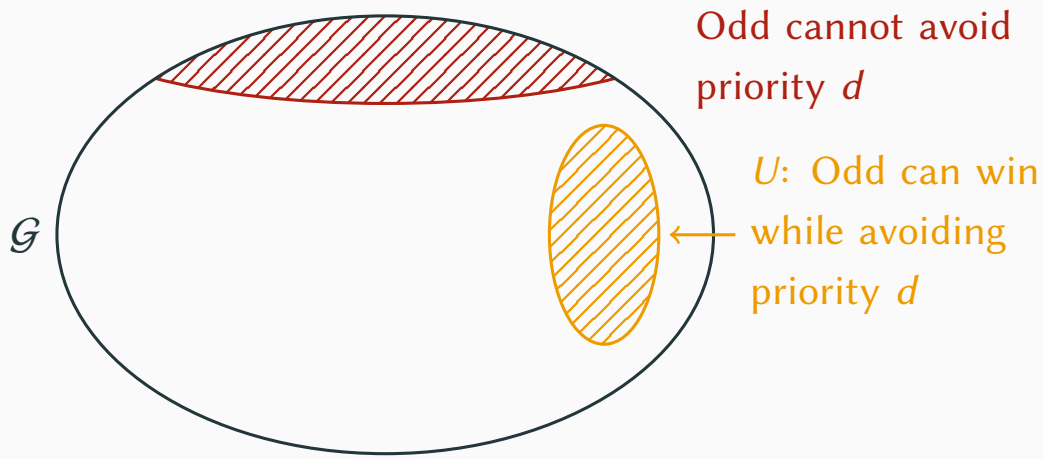

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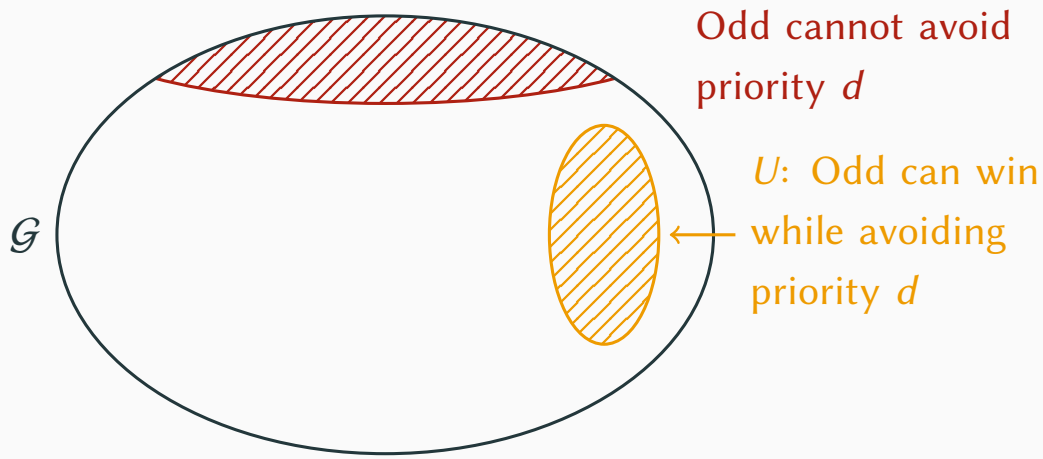

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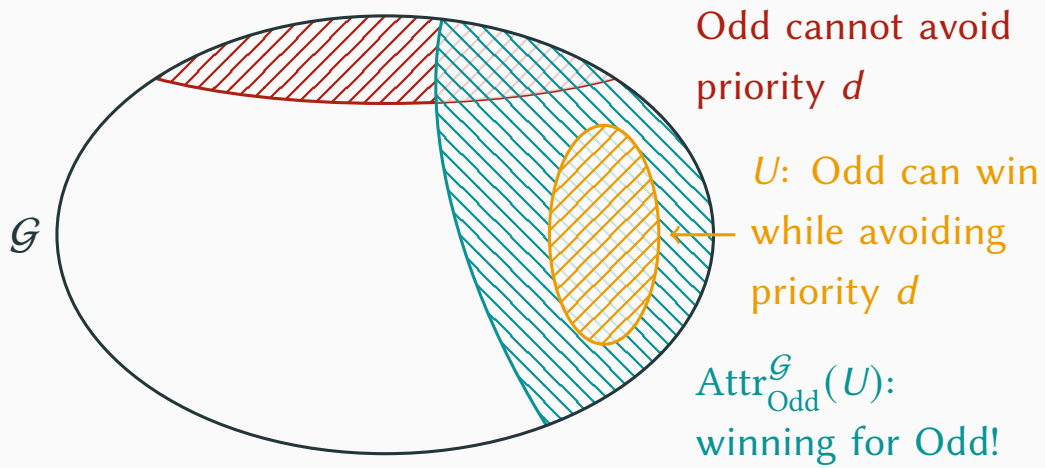

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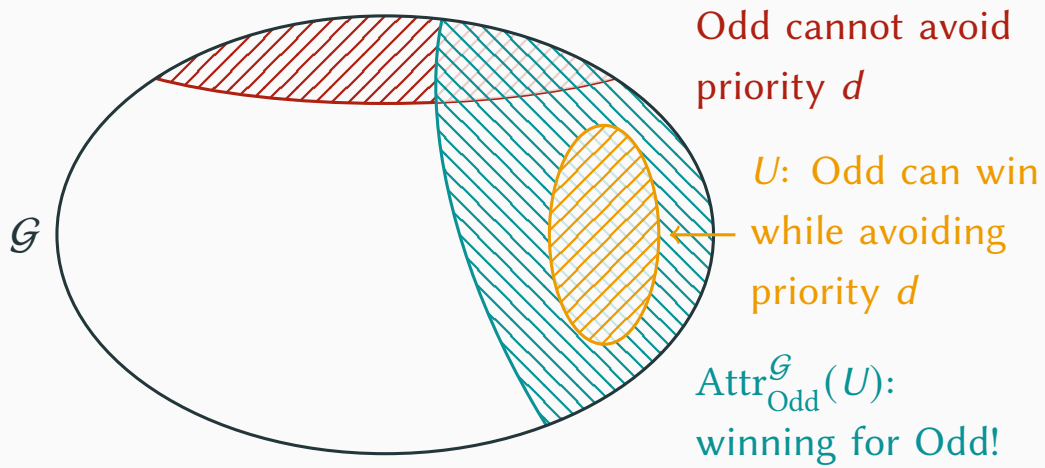

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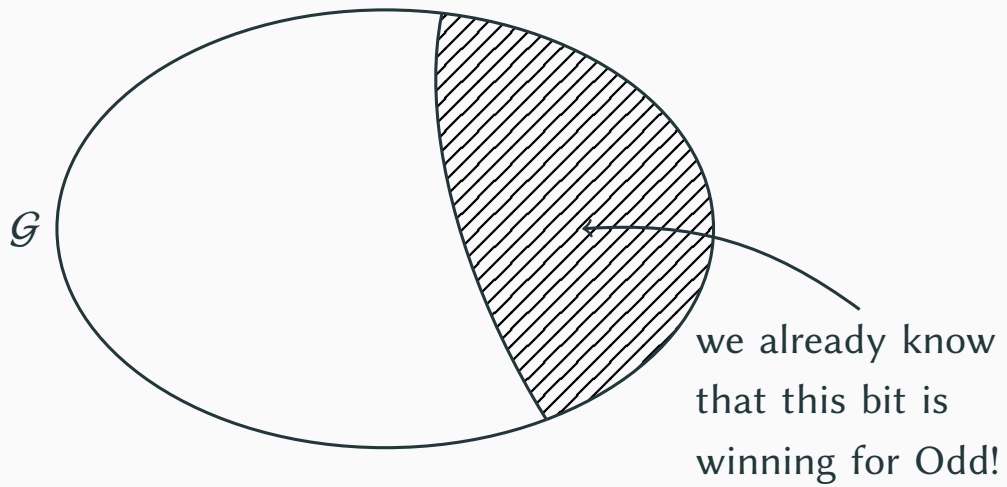

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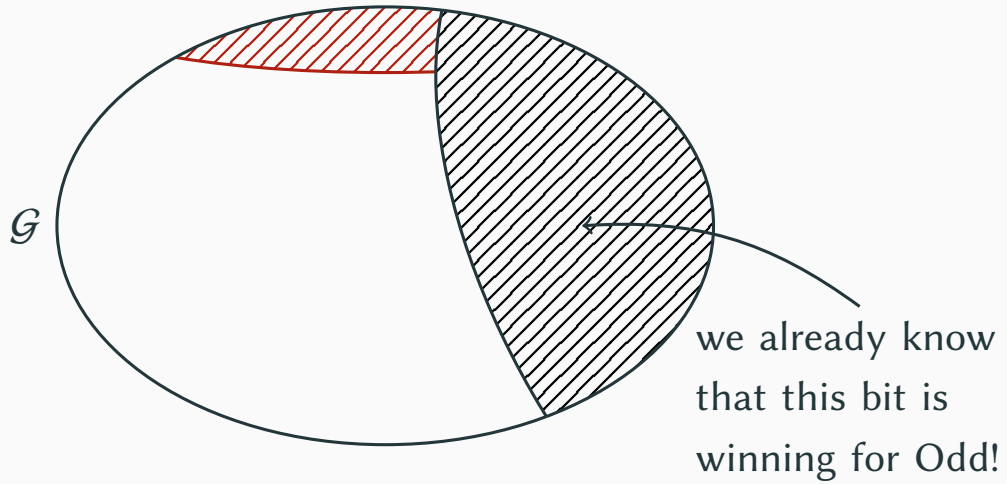

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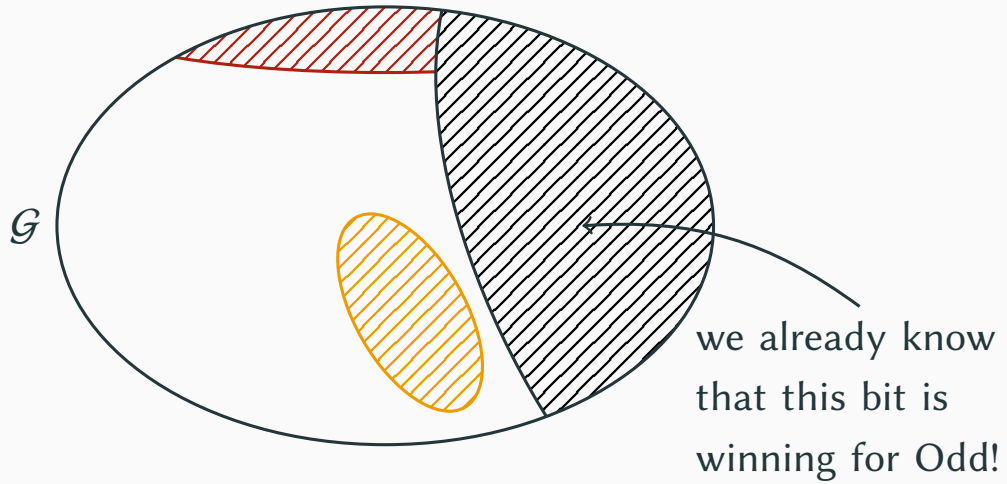

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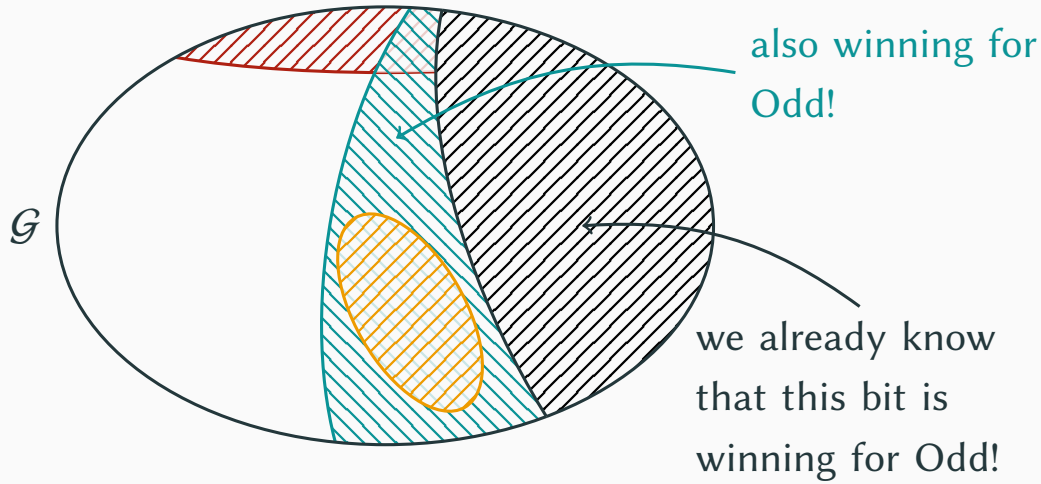

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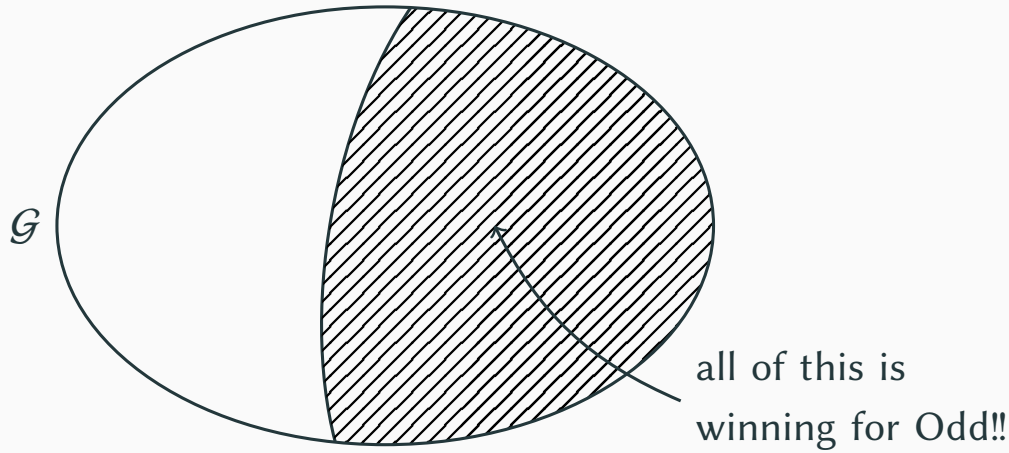

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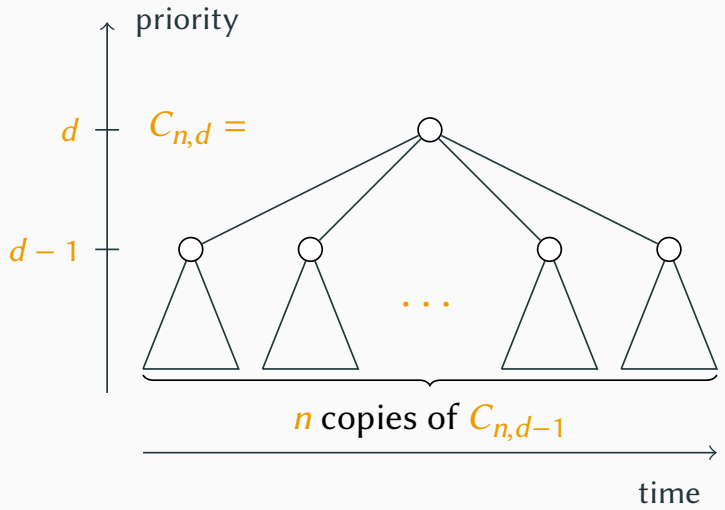

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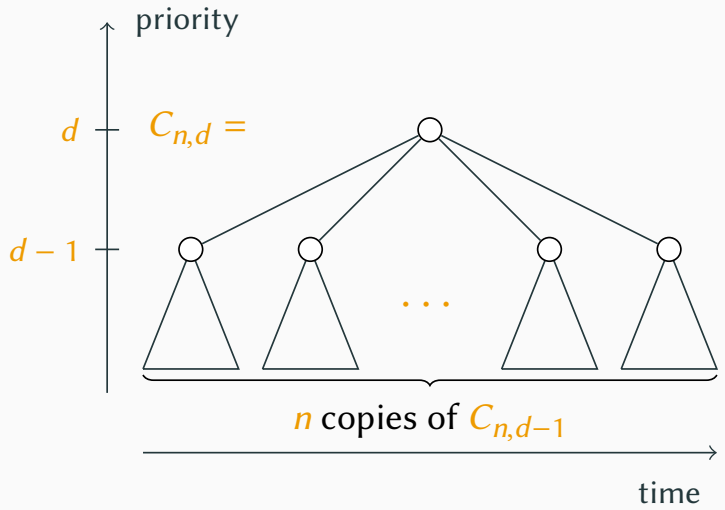

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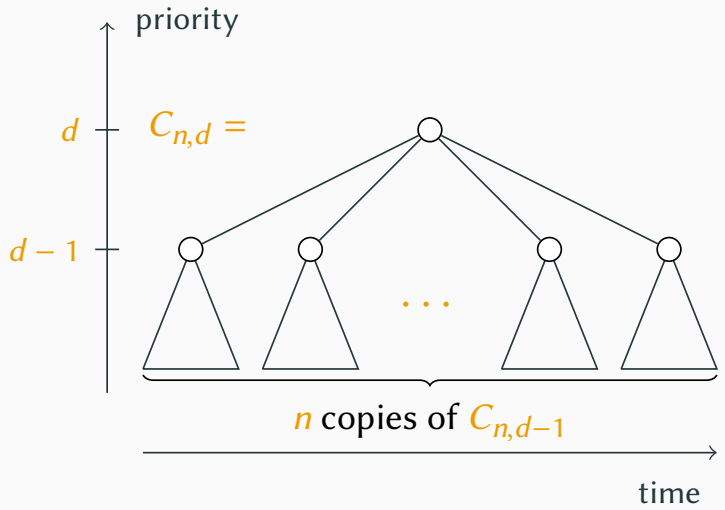

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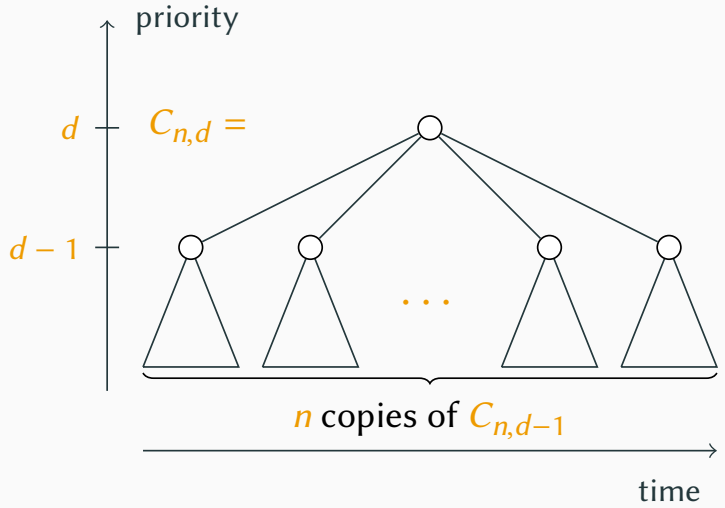

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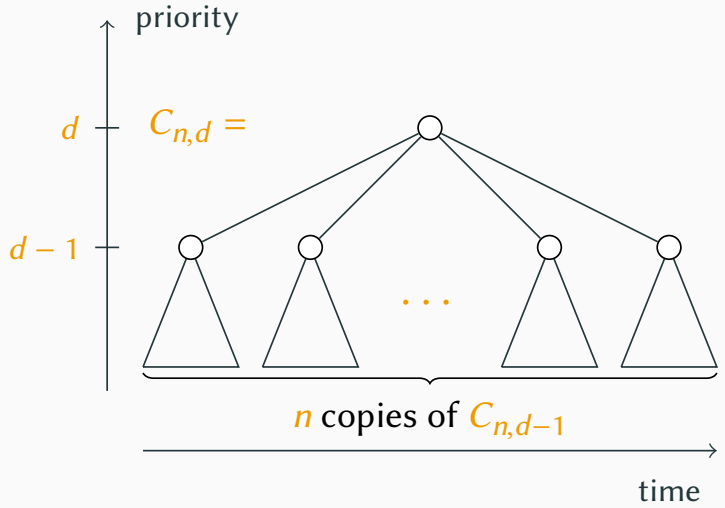

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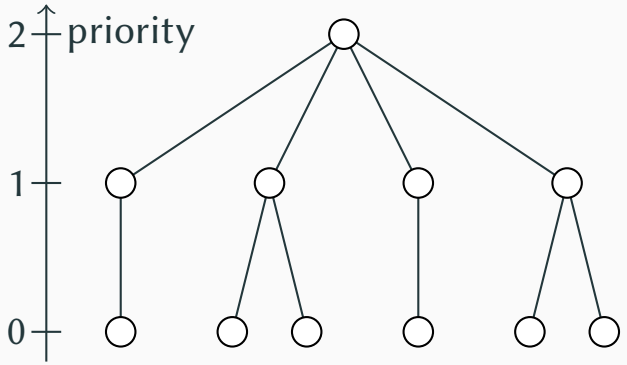

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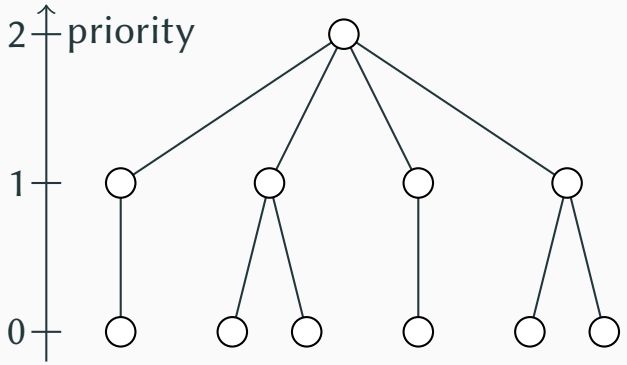

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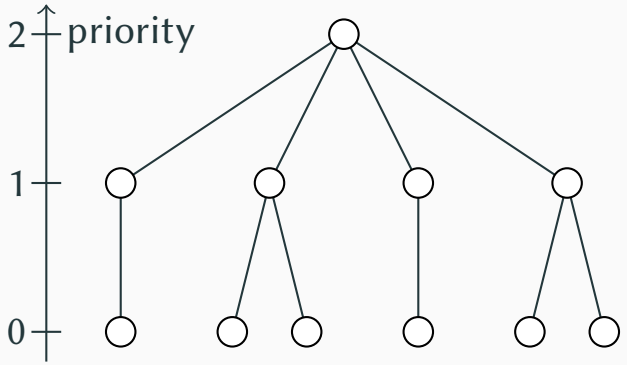

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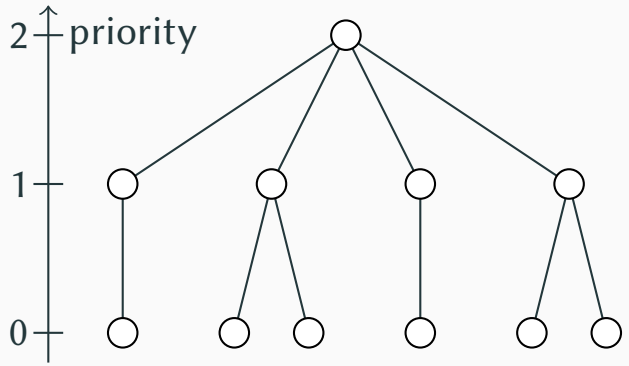

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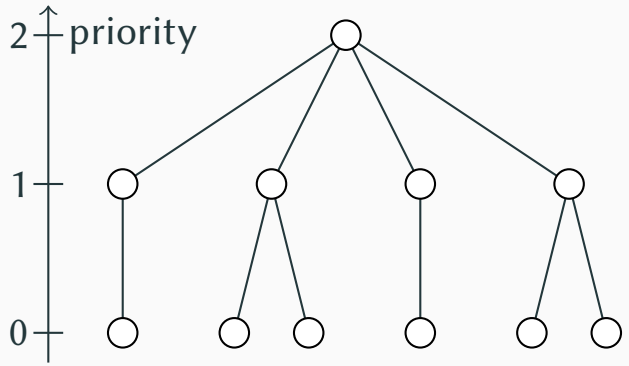

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   Iterate k times, where k is the number of children of the root of $\mathcal{T}$.

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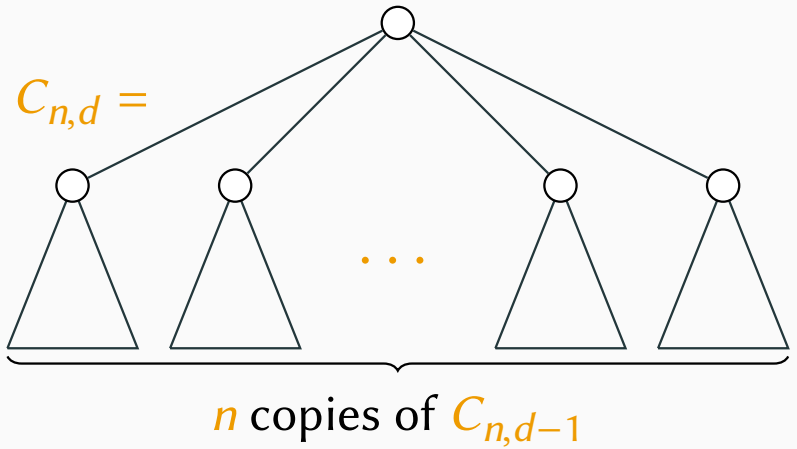

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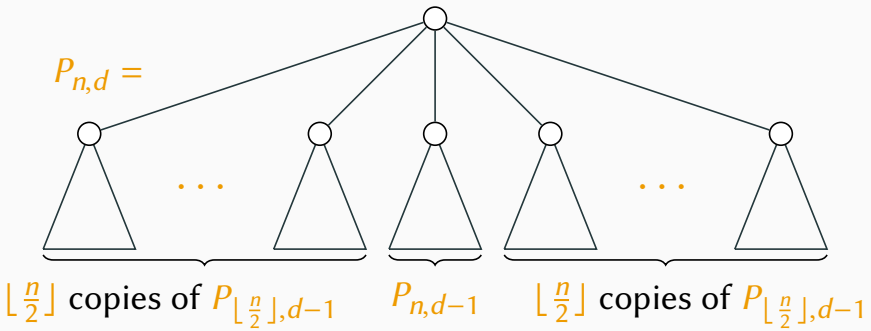

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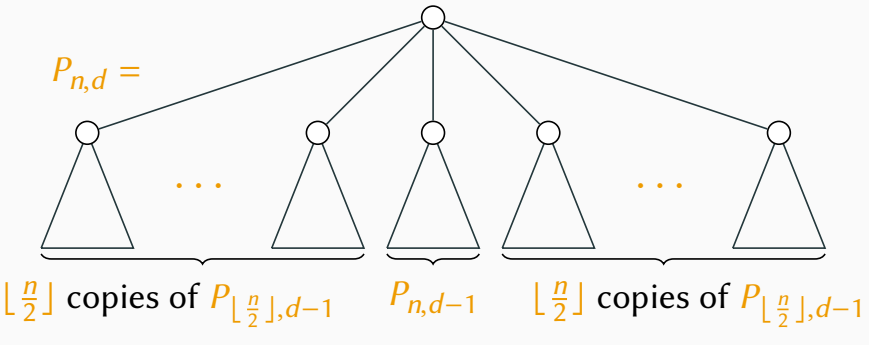

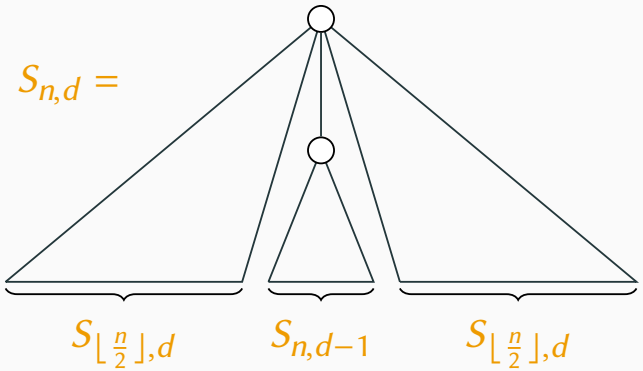

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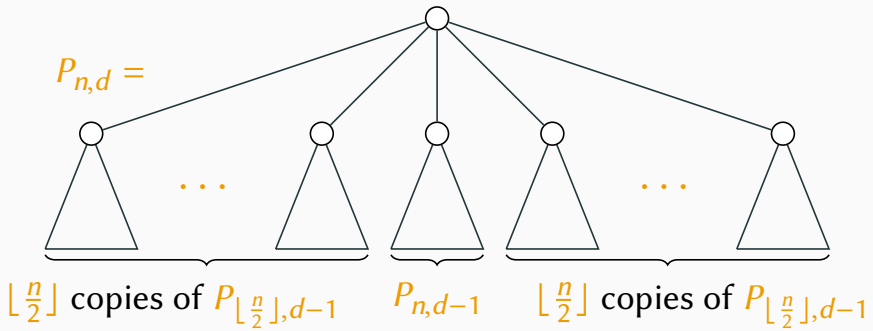

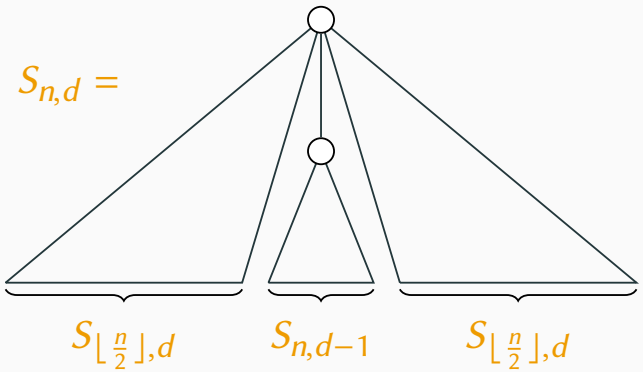

n: number of vertices & d: top priority

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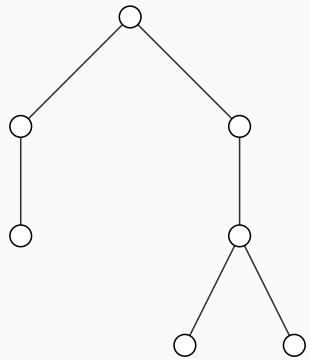
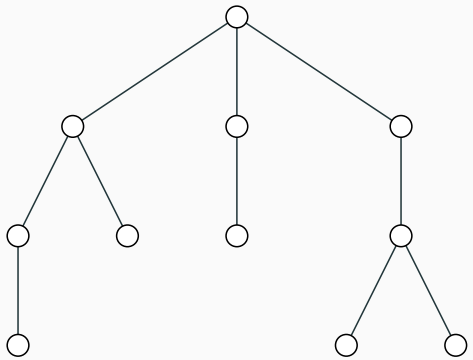
- Correctness: If $\mathcal{T}$ is big enough, then the algorithm is correct.

Ordered trees





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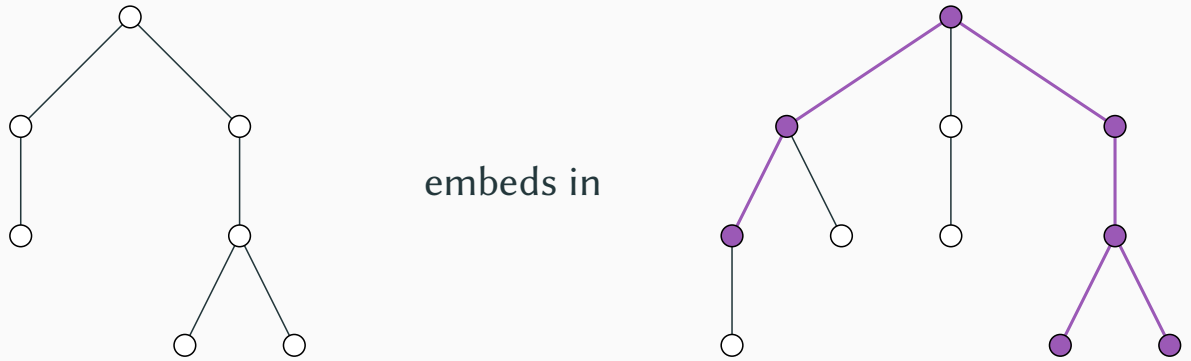
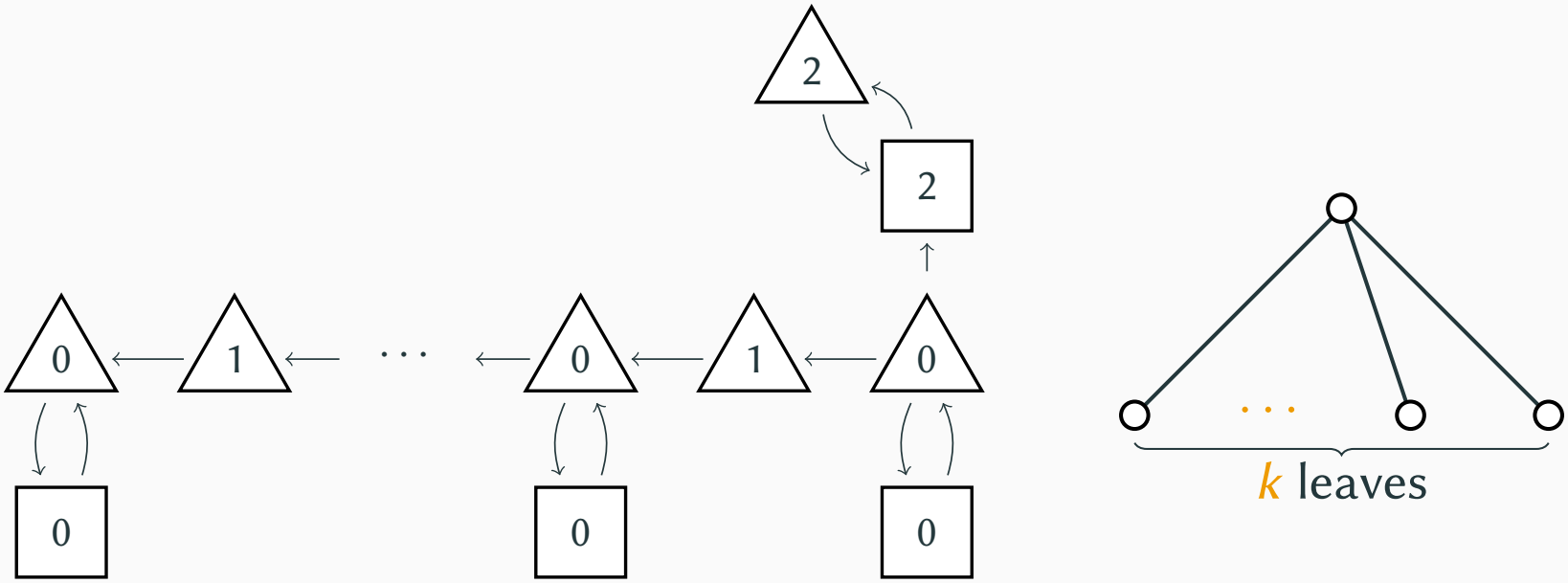


- Ordered trees: partially ordered by the "embedding" relation.
- Correctness: For every game $\mathcal{G}$, there exists a "small" tree $\mathcal{T}_\mathcal{G}$ such that the universal algorithm is correct whenever $\mathcal{T}_\mathcal{G}$ embeds in $\mathcal{T}$.

Attractor decomposition



An attractor decomposition $\mathrm{Win}_{\mathrm{Even}}(\mathcal{G})$ of $\mathcal{G}$ is a partition of the winning set describing the structure of Even's winning set.
[Daviaud-Jurdziński-Lehtinen, '18]

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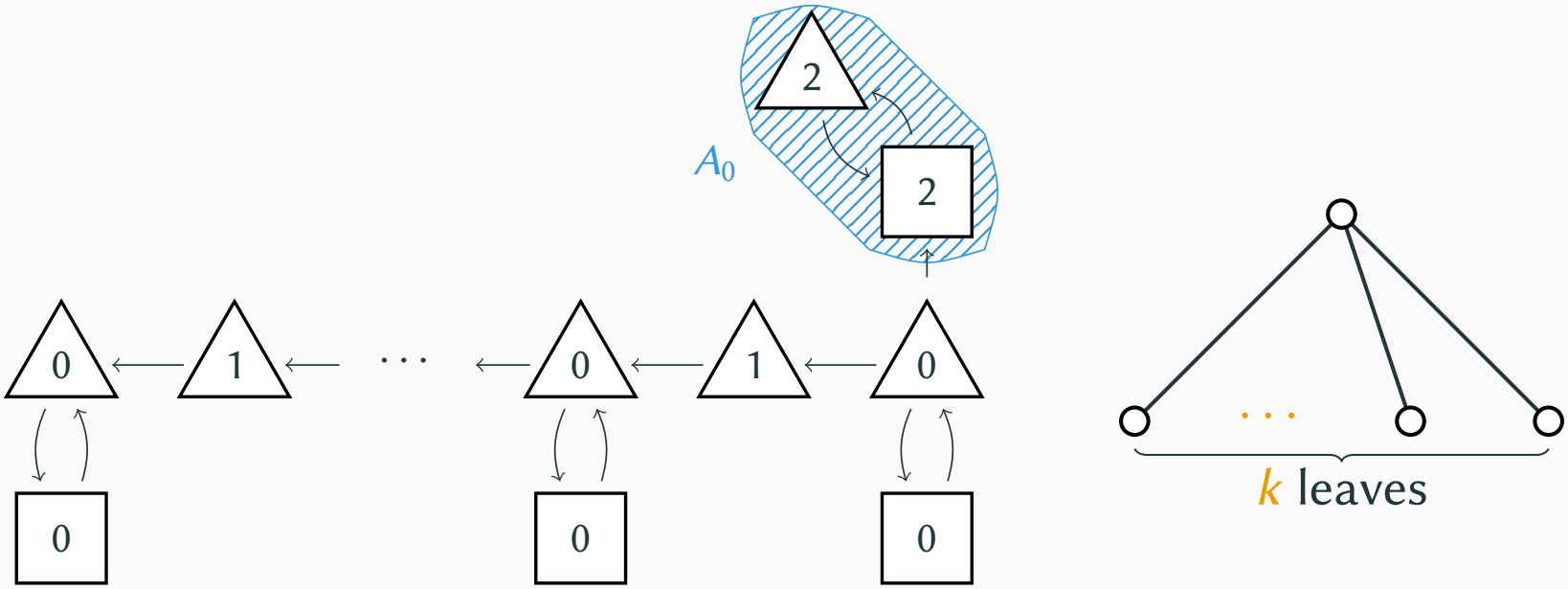

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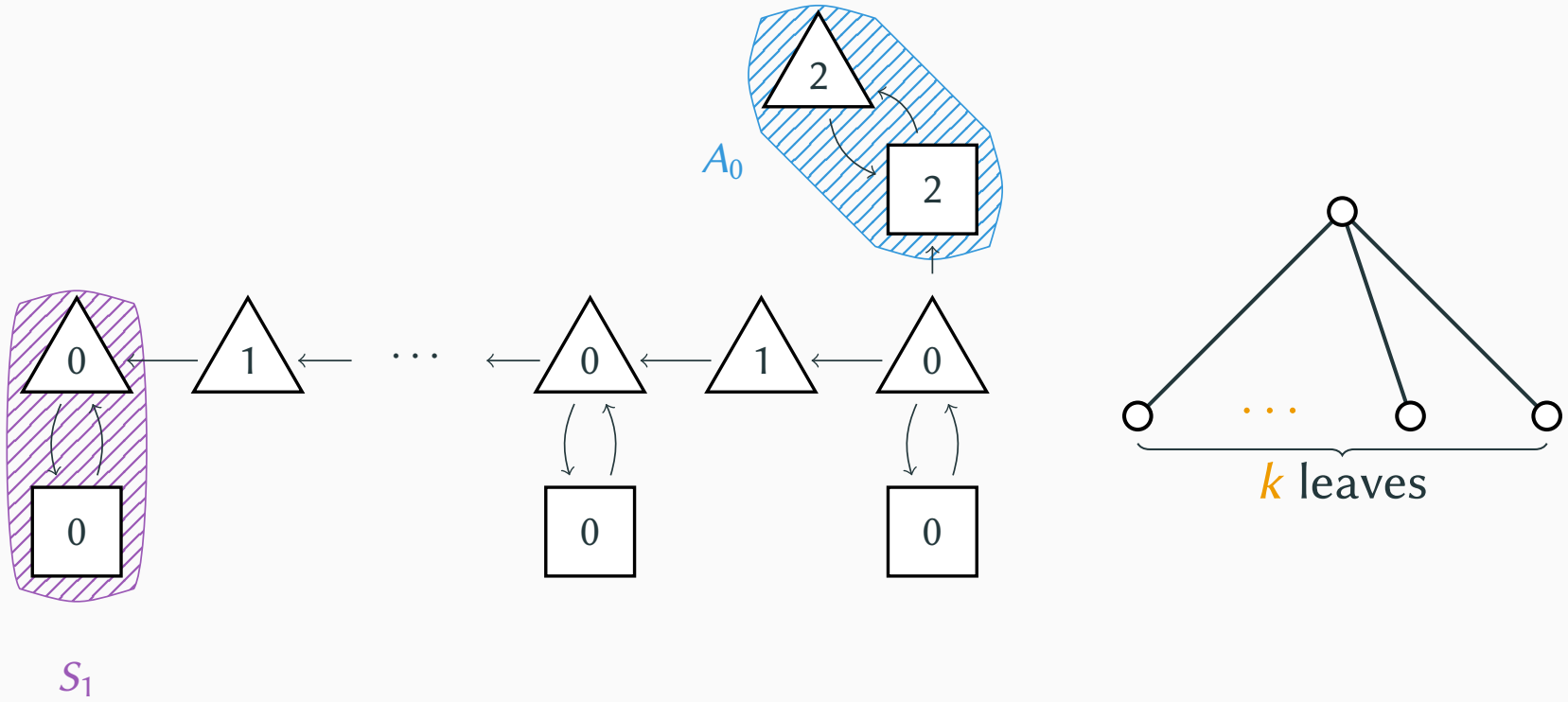

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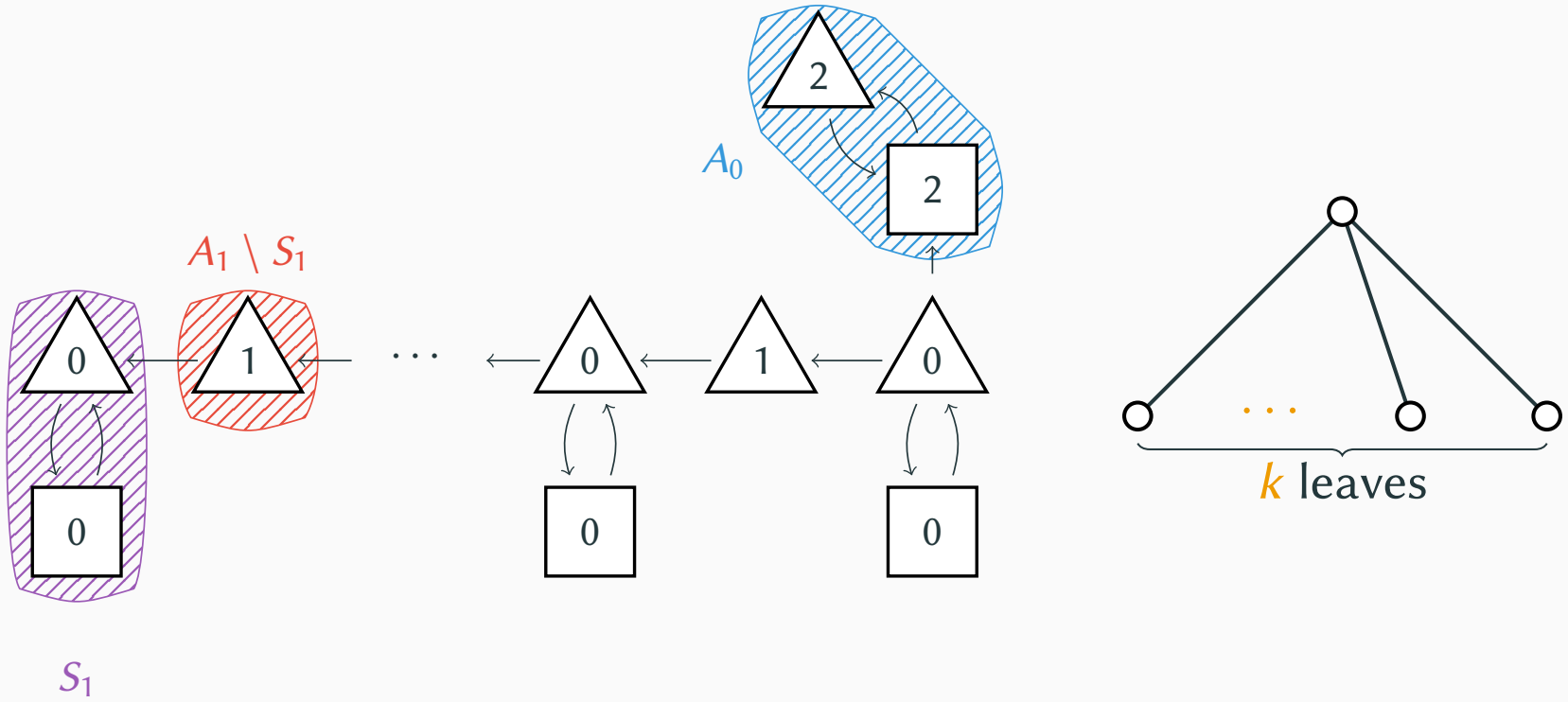

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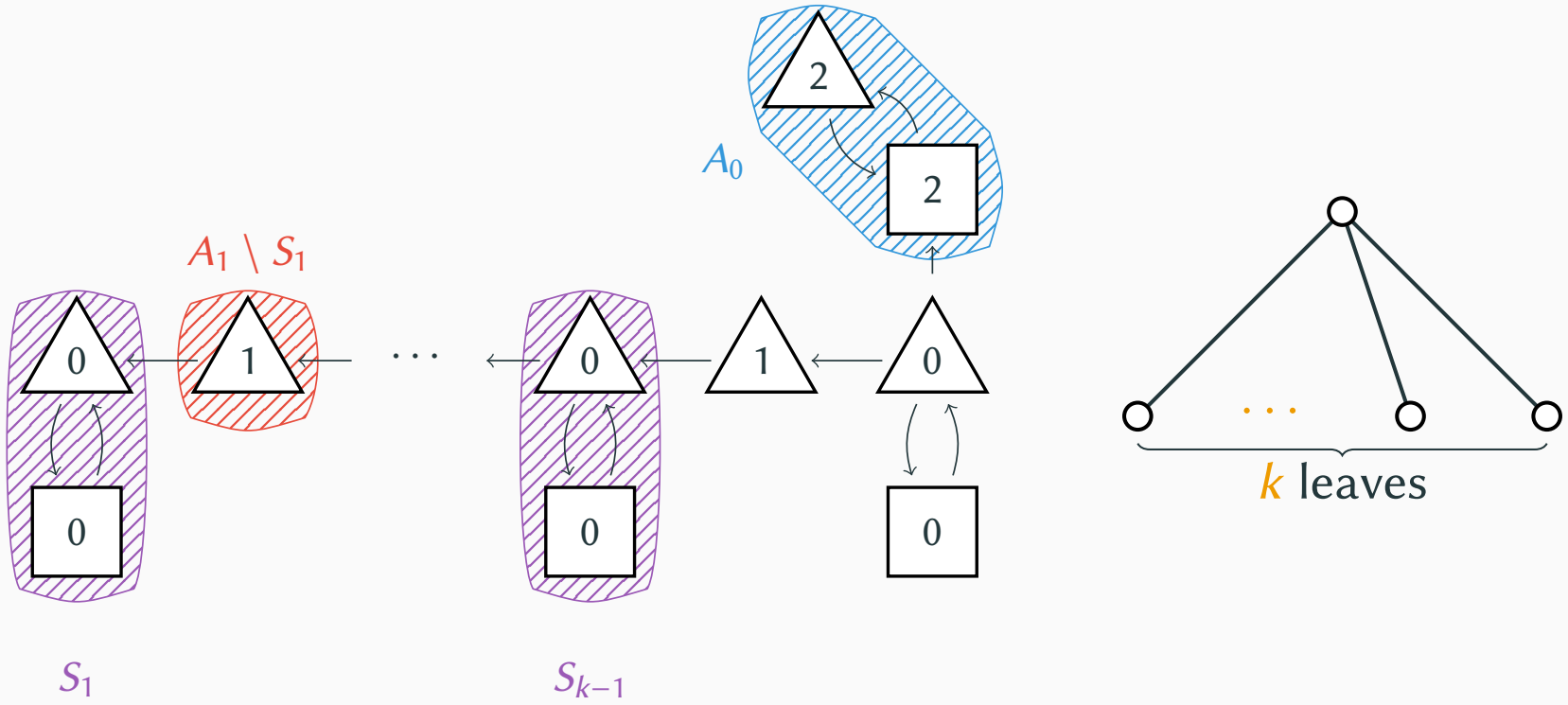

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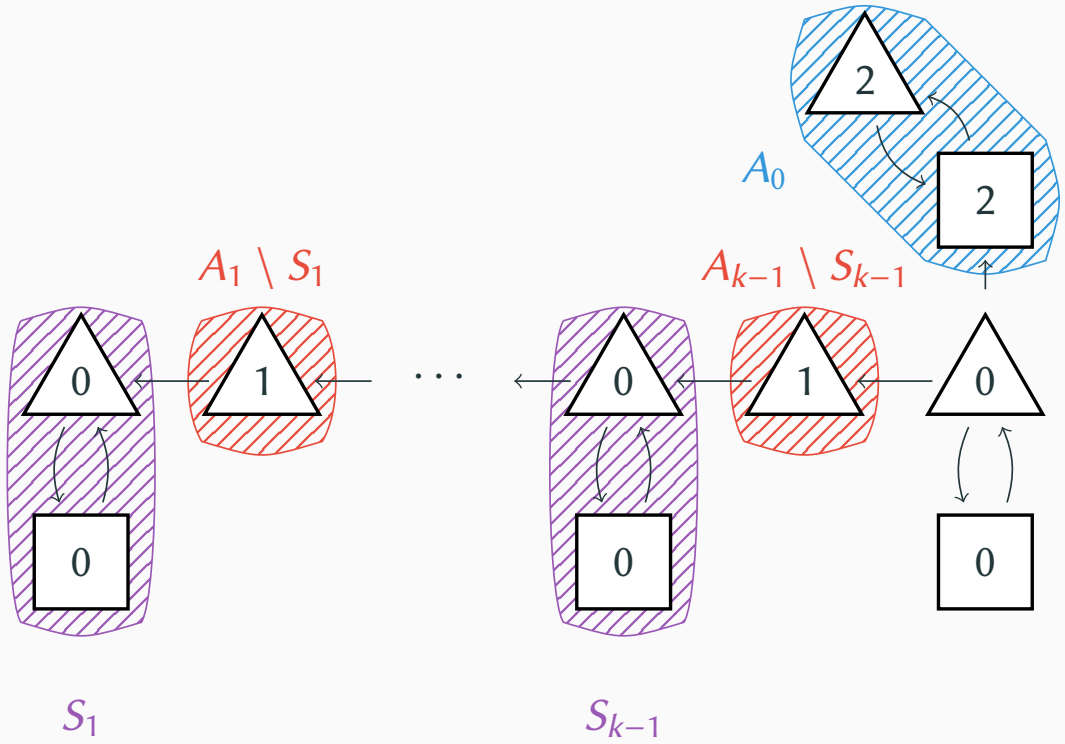

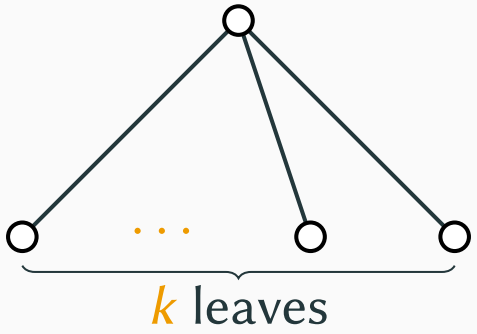

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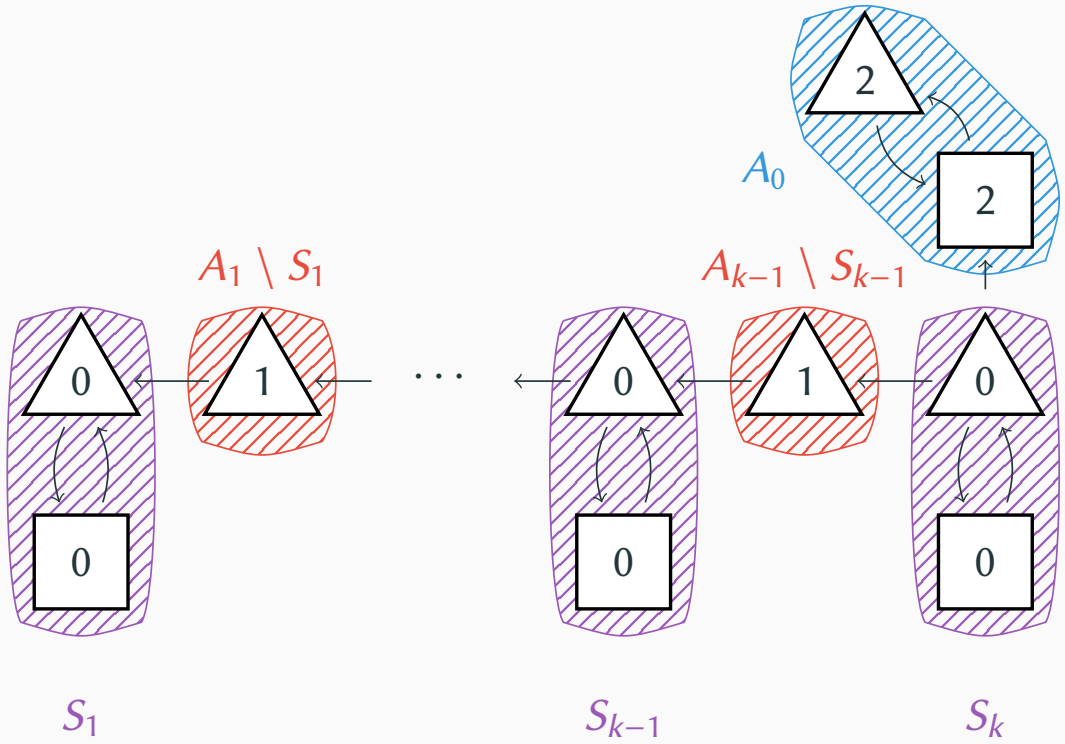

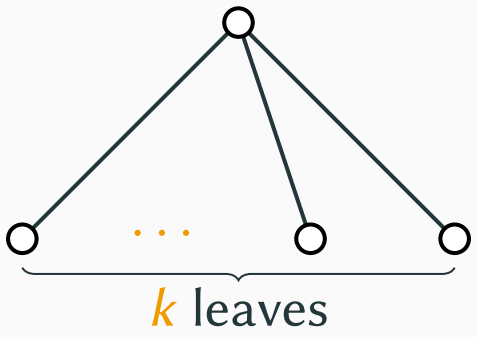

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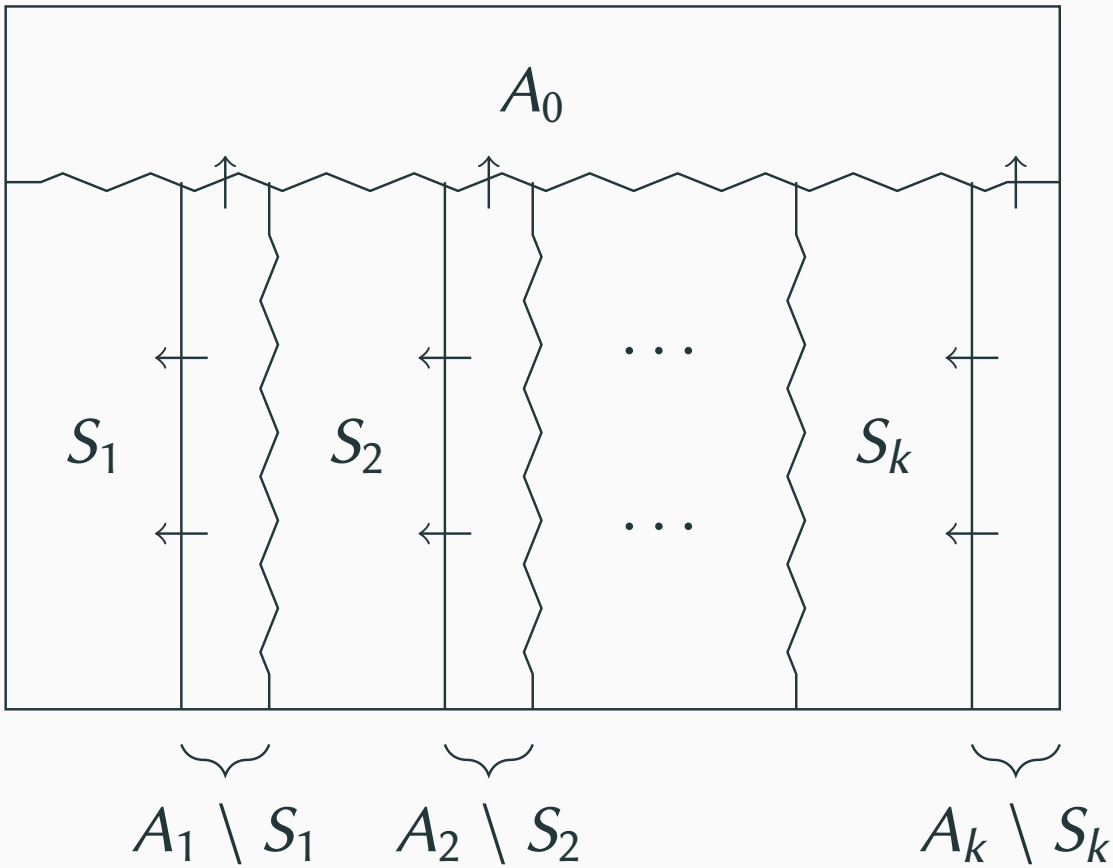

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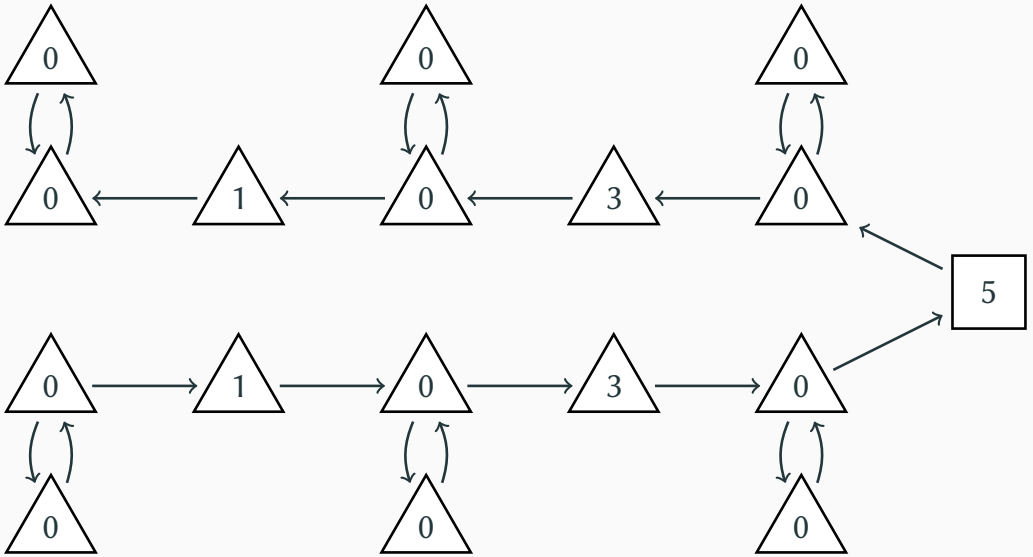

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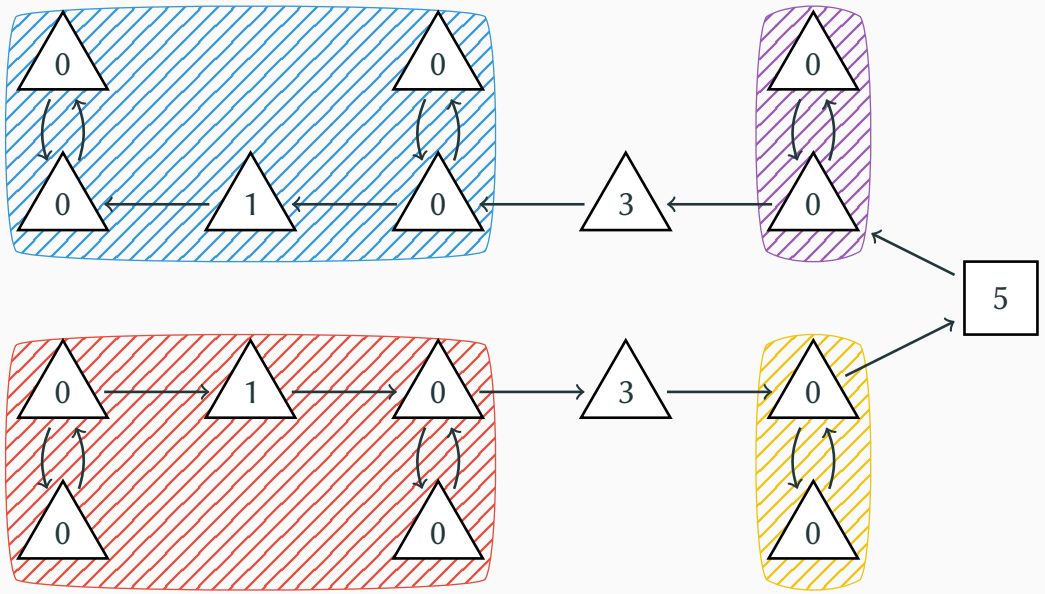

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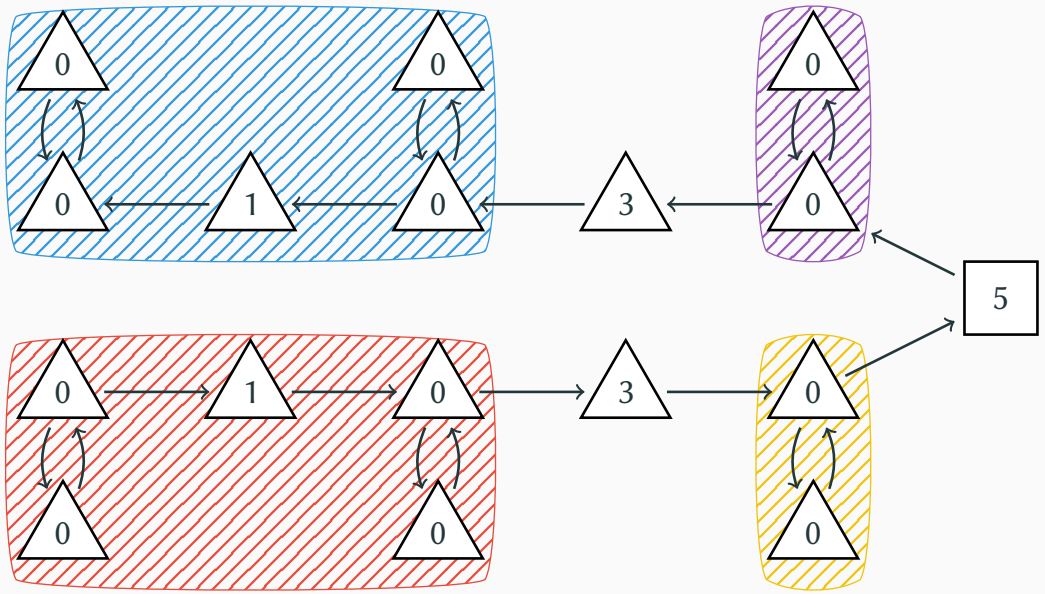

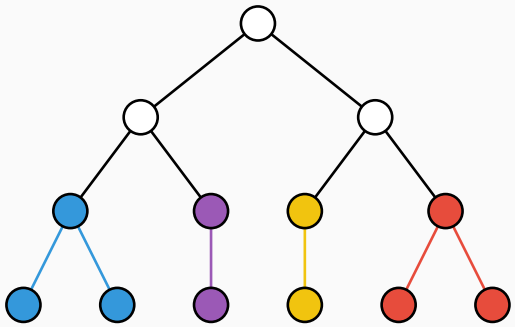

Embeddable decomposition theorem

Theorem: If D is subset of the winning set W for Even, if Odd can force the play to stay in D, for every attractor decomposition tree $\mathcal{T}_W$ of W, there exists an attractor decomposition tree $\mathcal{T}_D$ of D such that: $\mathcal{T}_D$ embeds in $\mathcal{T}_W$.

Embeddable decomposition theorem

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Attractor decomposition trees describe the shape of the structure of a winning region.

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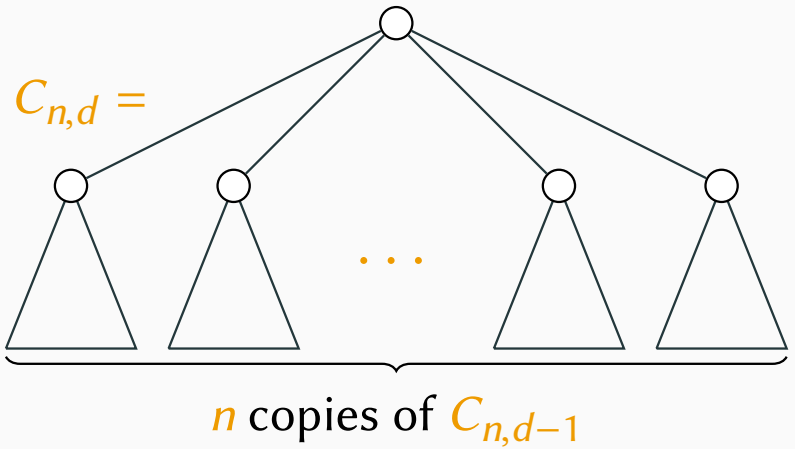

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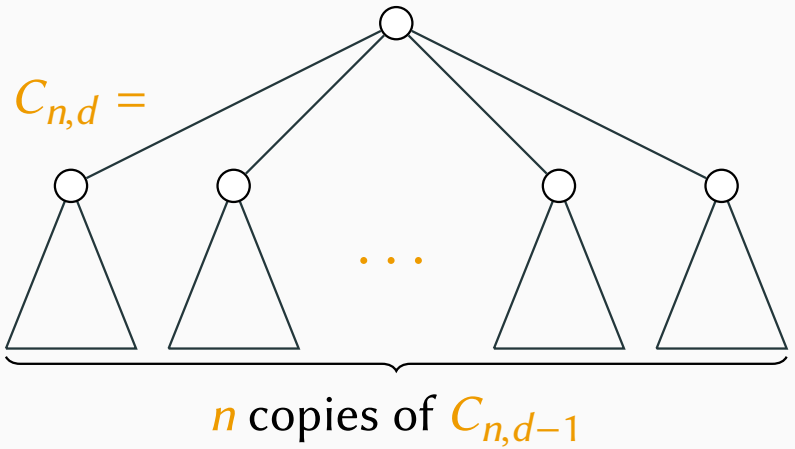

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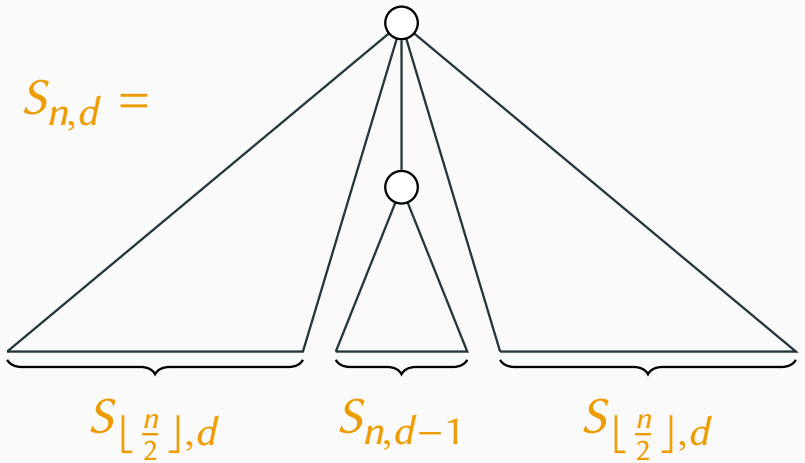

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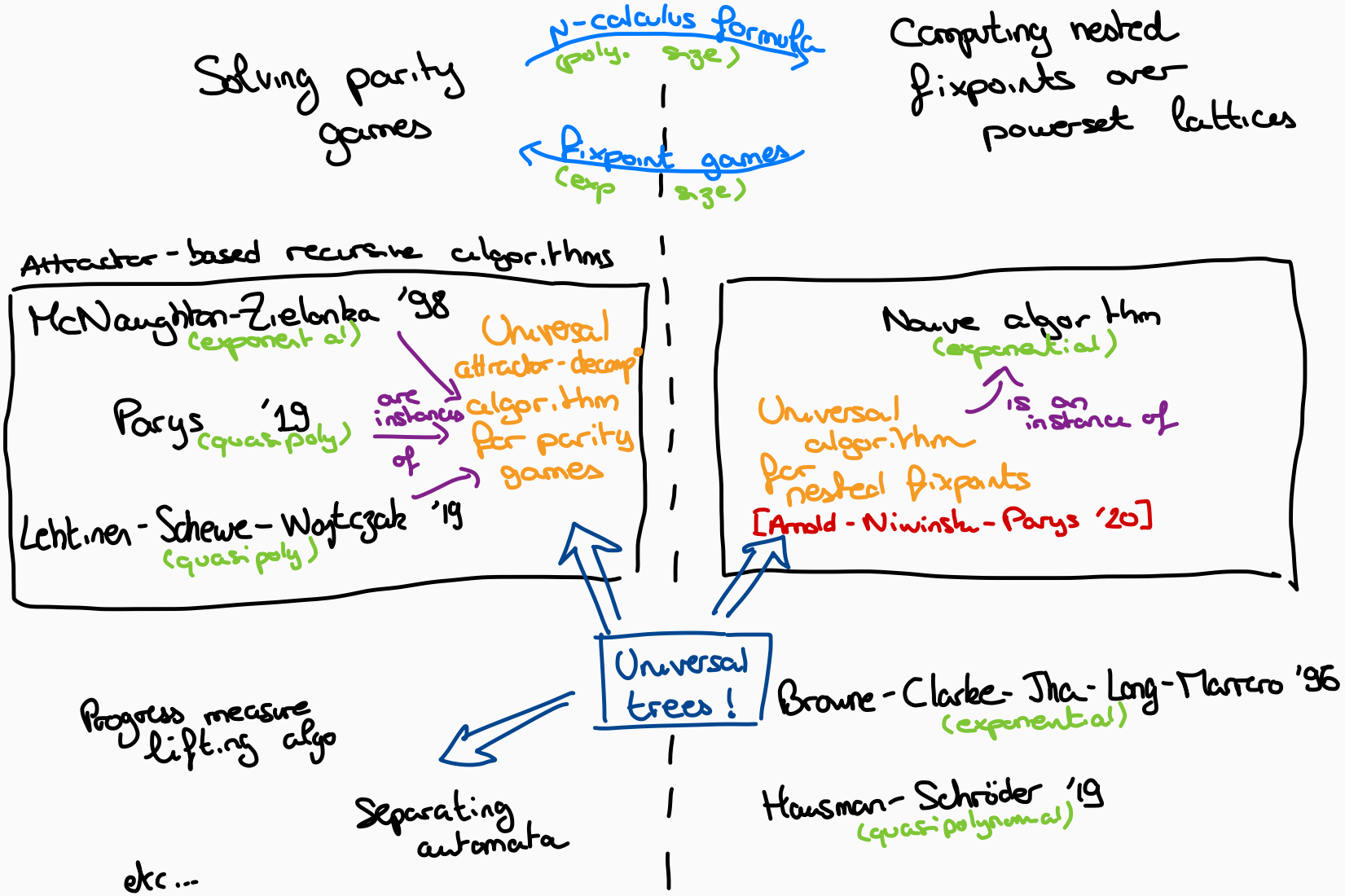
- Works if $\mathcal{T}_{\mathcal{G}}$ is the product of two universal trees.
- This applies to McNaughton-Zielonka '98, to Parys '19 and to Lehtinen-Schewe- Wojtczak '19.

Conclusion


Solving parity games
μ-calculus formula
(poly. size)
Fixpoint games
(exp size)
Computing nested fixpoints over powerset lattices
Attractor-based recursive algorithms
McNaughton-Zielonka '98
(exponential)
Parys '19
(quasi-poly)
Lehtinen-Schewe-Wojtczak '19
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are instances of
Universal attractor-decomp° algorithm for parity games
Naive algorithm
(exponential)
Universal algorithm for nested fixpoints
is an instance of
[Arnold-Niwinski-Parys '20]
Universal trees!
Browne-Clarke-Jha-Long-Marrero '96
(exponential)
Progress measure lifting algo
Separating automata
Hausmann-Schröder '19
(quasi-polynomial)
etc...